Appendix for a paper appearing in Journal of Functional Programming **14** (3): 317–363, May 2004, Cambridge University Press.

Proof appendix:
Composition of functions with accumulating parameters

JANIS VOIGTLÄNDER∗ and ARMIN KÜHNEMANN
Department of Computer Science, Dresden University of Technology,
D-01062 Dresden, Germany
(*e-mail:* {voigt,kuehne}@tcs.inf.tu-dresden.de)

Abstract

In this appendix to the article "Composition of functions with accumulating parameters" we prove Theorem 5.2 of that paper, showing that Construction 5.1 produces an mtt that is equivalent to the composition of the two given ones. Firstly, we will formalise the idea of "walking upwards" in the intermediate result to obtain the context parameters of calls of the second mtt's states on the context parameters of the first mtt, as presented in Subsection 4.6. To that purpose, we introduce functions that—as we will prove—give answers to the question **Q** in Subsection 4.5. We will introduce some auxiliary notions, in particular an ordering relation that will be useful to prove the "cutting" of potential cycles mentioned in Subsection 4.10. We present some properties of the mentioned functions and relations, and prepare the main proof by establishing some necessary technical lemmata.

In the following, let M_1 and M_2 be two fixed mtts as in Construction 5.1. We will also use the notations and names introduced there. For technical reasons, we assume without loss of generality that there are no name conflicts between M_1 and M_2, i.e., all involved ranked alphabets are pairwise disjoint. This can always be achieved by renaming and guarantees, e.g., that rewrite rules of M_1 and M_2 can be applied in arbitrary order ($\Rightarrow_{R_1 \cup R_2}$ is confluent).

As noted below Theorem 5.2, we could generalise the weakly single-use property in Definition 3.6 by dropping condition (ii) and requiring condition (i) only for states g that do have context parameters, without requiring any change to Construction 5.1. In fact, the proofs in this appendix will from Definition 3.6 only use condition (i) and this only for states of M_2 with rank greater than one. Also, the non-copying restriction of M_1 and the weakly single-use restriction of M_2 will not be needed if one of the two is a tdtt. Thus, our correctness proof for Construction 5.1 also incorporates proofs for the known results $TOP ; MAC \subseteq MAC$ (Engelfriet, 1981) and $MAC ; TOP \subseteq MAC$ (Engelfriet & Vogler, 1985).

Firstly, we introduce functions that can be used to answer question **Q** from Subsection 4.5.

∗ This author was supported by the DFG under grant KU 1290/2-1.

Definition A.1 ($\overline{par}_{f,t}$*- and* $\overline{par}_{\phi,(t_1,\ldots,t_p)}$*-functions*)
The functions in the set:

$$\begin{aligned}&\{\overline{par}_{f,t} : \{(k,g,l) \mid k \in [r], g \in G^{(s+1)}, l \in [s]\} \to RHS(G,\Omega,Y_r,Z_G)\\ &\quad \mid f \in F^{(r+1)}, t \in T_\Sigma\}\\ \cup\, &\{\overline{par}_{\phi,(t_1,\ldots,t_p)} : \{(\pi,g',l) \mid \pi \in paths(\phi), lab(\phi,\pi) \notin U_p, g' \in G^{(s'+1)}, l \in [s']\}\\ &\qquad \to RHS(G,\Omega,Y_r,Z_G)\\ &\quad \mid (r+1) \in rank(F), r > 0, p \in \mathbb{N}, \phi \in RHS(F,\Delta,U_p,Y_r), t_1,\ldots,t_p \in T_\Sigma\}\end{aligned}$$

are defined by mutual recursion as follows.

For every $f \in F^{(r+1)}$, $g \in G^{(s+1)}$, $k \in [r]$, $l \in [s]$, $\sigma \in \Sigma^{(p)}$ and $t_1,\ldots,t_p \in T_\Sigma$: if $rhs_{f,\sigma}$ does not contain the context variable y_k, then

$$\overline{par}_{f,\sigma(t_1,\ldots,t_p)}(k,g,l) = nil\,;$$

otherwise

$$\overline{par}_{f,\sigma(t_1,\ldots,t_p)}(k,g,l) = \overline{par}_{rhs_{f,\sigma},(t_1,\ldots,t_p)}(\pi_{y_k},g,l)\,,$$

where $\pi_{y_k} \in paths(rhs_{f,\sigma})$ is the unique path with $lab(rhs_{f,\sigma},\pi_{y_k}) = y_k$ (notice that M_1 is non-copying).

For every $(r+1) \in rank(F)$ with $r > 0$, $p \in \mathbb{N}$, $\phi \in RHS(F,\Delta,U_p,Y_r)$ and $t_1,\ldots,t_p \in T_\Sigma$, the function $\overline{par}_{\phi,(t_1,\ldots,t_p)}$ is defined by induction on the prefix-order of paths in ϕ. For every $g' \in G^{(s'+1)}$ and $l \in [s']$:

- $\overline{par}_{\phi,(t_1,\ldots,t_p)}(\varepsilon,g',l) = z_{g',l}$
- For every $j \in \mathbb{N}_+$ and $\pi j \in paths(\phi)$ with $lab(\phi,\pi j) \notin U_p$, the definition is by case distinction on $lab(\phi,\pi)$ as follows:

  $\underline{lab(\phi,\pi) = \delta}$ for some $\delta \in \Delta^{(q)}$ and $j \in [q]$:

  If, with $g'' \in G^{(s''+1)}$ and $\psi_1,\ldots,\psi_{s'} \in RHS(G,\Omega,V_q,Z_{s''})$, the only occurrence of a $g'(v_j,\ldots)$-call in the δ-rules of the weakly single-use mtt M_2 looks as follows:

  $$g''(\delta(v_1,\ldots,v_q),z_1,\ldots,z_{s''}) \to \ldots g'(v_j,\psi_1,\ldots,\psi_{s'})\ldots\ ,$$

  then:

  $$\begin{aligned}&\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi j,g',l)\\ &= nf(\Rightarrow_{R_2},\psi_l[v_d \leftarrow nf(\Rightarrow_{R_1},sub(\phi,\pi d)[u_c \leftarrow t_c, c \in [p]]), d \in [q],\\ &\qquad\qquad z_m \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi,g'',m), m \in [s'']]).\end{aligned}$$

  If no such call exists in the δ-rules of M_2, then $\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi j,g',l) = nil$.

  $\underline{lab(\phi,\pi) = f}$ for some $f \in F^{(q+1)}$, $1 \leq j-1 \leq q$ and $lab(\phi,\pi 1) = u_i \in U_p$:

  $$\begin{aligned}&\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi j,g',l)\\ &= nf(\Rightarrow_{R_2},\overline{par}_{f,t_i}(j-1,g',l)\\ &\qquad [y_b \leftarrow nf(\Rightarrow_{R_1},sub(\phi,\pi(b+1))[u_c \leftarrow t_c, c \in [p]]), b \in [q],\\ &\qquad\ z_{g'',m} \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi,g'',m),\ g'' \in G^{(s''+1)}, m \in [s'']]).\end{aligned}$$ $\diamond$

Note that the terms produced by $\overline{par}_{f,t}$- and $\overline{par}_{\phi,(t_1,\dots,t_p)}$-functions, respectively, are always $\Rightarrow_{R_2}$-normal forms. Also, note the similarities between the definitions of $\overline{par}_{f,t}$-functions and right-hand sides for (k_f, l_g)-states, and between the definitions of $\overline{par}_{\phi,(t_1,\dots,t_p)}$- and par_ϕ-functions, respectively. These correspondences will be made precise in the statements **I**b and **II**b of Lemma A.15.

Example A.2 ($\overline{par}_{f,t}$*- and* $\overline{par}_{\phi,(t_1,\dots,t_p)}$*-functions*)
We refer to the mtts M_1 and M_2 from Example 4.2 and compute as follows:

$$
\begin{array}{l}
\overline{par}_{f_1,A(B(E))}(1, g_2, 1)\\
\quad = \text{ (by definition of } \overline{par}_{f_1,A(B(E))}\text{, with } rhs_{f_1,A} = \alpha(f_1(u_1, \underline{y_1})))\\
\overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(12, g_2, 1)\\
\quad = \text{ (by definition of } \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))})\\
nf(\Rightarrow_{R_2}, \overline{par}_{f_1,B(E)}(1, g_2, 1)[y_1 \leftarrow nf(\Rightarrow_{R_1}, y_1[u_1 \leftarrow B(E)]),\\
\qquad z_{g_1,1} \leftarrow \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_1, 1),\\
\qquad z_{g_2,1} \leftarrow \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_2, 1)])\\
\quad = \text{ (by definition of } \overline{par}_{f_1,B(E)}\text{, with } rhs_{f_1,B} = f_1(u_1, \beta(\underline{y_1})))\\
nf(\Rightarrow_{R_2}, \overline{par}_{f_1(u_1,\beta(y_1)),(E)}(21, g_2, 1)[z_{g_1,1} \leftarrow \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_1, 1),\\
\qquad z_{g_2,1} \leftarrow \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_2, 1)])\\
\quad = \text{ (by definition of } \overline{par}_{f_1(u_1,\beta(y_1)),(E)},\\
\qquad \text{with } g_2(\beta(v_1), z_1) \rightarrow \underline{g_2(v_1, \omega(g_1(v_1, z_1), z_1))} \in R_2)\\
nf(\Rightarrow_{R_2}, nf(\Rightarrow_{R_2}, \omega(g_1(v_1, z_1), z_1)[v_1 \leftarrow nf(\Rightarrow_{R_1}, y_1[u_1 \leftarrow E]),\\
\qquad z_1 \leftarrow \overline{par}_{f_1(u_1,\beta(y_1)),(E)}(2, g_2, 1)])\\
\qquad [z_{g_1,1}, z_{g_2,1} \leftarrow \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_1, 1), \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_2, 1)])\\
\quad = \text{ (since } nf(\Rightarrow_{R_1}, y_1[u_1 \leftarrow E]) = y_1 \text{ and by definition of } \overline{par}_{f_1(u_1,\beta(y_1)),(E)})\\
nf(\Rightarrow_{R_2}, nf(\Rightarrow_{R_2}, \omega(g_1(y_1, z_1), z_1)\\
\qquad [z_1 \leftarrow nf(\Rightarrow_{R_2}, \overline{par}_{f_1,E}(1, g_2, 1)[y_1 \leftarrow nf(\Rightarrow_{R_1}, \beta(y_1)[u_1 \leftarrow E]),\\
\qquad\qquad z_{g_1,1} \leftarrow \overline{par}_{f_1(u_1,\beta(y_1)),(E)}(\varepsilon, g_1, 1),\\
\qquad\qquad z_{g_2,1} \leftarrow \overline{par}_{f_1(u_1,\beta(y_1)),(E)}(\varepsilon, g_2, 1)])])\\
\qquad [z_{g_1,1}, z_{g_2,1} \leftarrow \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_1, 1), \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_2, 1)])\\
\quad = \text{ (by definition of } \overline{par}_{f_1(u_1,\beta(y_1)),(E)})\\
nf(\Rightarrow_{R_2}, nf(\Rightarrow_{R_2}, \omega(g_1(y_1, z_1), z_1)[z_1 \leftarrow nf(\Rightarrow_{R_2}, \overline{par}_{f_1,E}(1, g_2, 1)[y_1 \leftarrow \beta(y_1)])])\\
\qquad [z_{g_1,1}, z_{g_2,1} \leftarrow \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_1, 1), \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_2, 1)])\\
\quad = \text{ (by definition of } \overline{par}_{f_1,E}\text{, with } rhs_{f_1,E} = \underline{y_1})\\
nf(\Rightarrow_{R_2}, nf(\Rightarrow_{R_2}, \omega(g_1(y_1, z_1), z_1)[z_1 \leftarrow nf(\Rightarrow_{R_2}, \overline{par}_{y_1,()}(\varepsilon, g_2, 1)[y_1 \leftarrow \beta(y_1)])])\\
\qquad [z_{g_1,1}, z_{g_2,1} \leftarrow \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_1, 1), \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_2, 1)])\\
\quad = \text{ (by definition of } \overline{par}_{y_1,()})\\
nf(\Rightarrow_{R_2}, nf(\Rightarrow_{R_2}, \omega(g_1(y_1, z_1), z_1)[z_1 \leftarrow nf(\Rightarrow_{R_2}, z_{g_2,1}[y_1 \leftarrow \beta(y_1)])])\\
\qquad [z_{g_1,1}, z_{g_2,1} \leftarrow \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_1, 1), \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_2, 1)])\\
\quad = \text{ (since } nf(\Rightarrow_{R_2}, z_{g_2,1}[y_1 \leftarrow \beta(y_1)]) = z_{g_2,1})\\
nf(\Rightarrow_{R_2}, \omega(g_1(y_1, z_{g_2,1}), z_{g_2,1})[z_{g_1,1} \leftarrow \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_1, 1),\\
\qquad z_{g_2,1} \leftarrow \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(1, g_2, 1)])\\
\quad = \text{ (by definition of } \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))},
\end{array}
$$

$$\text{with } g_1(\alpha(v_1), z_1) \to g_1(v_1, \underline{g_2(v_1, z_1)}) \in R_2)$$

$$nf(\Rightarrow_{R_2}, \omega(g_1(y_1, z_{g_2,1}), z_{g_2,1})$$
$$[z_{g_2,1} \leftarrow nf(\Rightarrow_{R_2}, z_1[v_1 \leftarrow nf(\Rightarrow_{R_1}, f_1(u_1, y_1)[u_1 \leftarrow B(E)]),$$
$$z_1 \leftarrow \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))}(\varepsilon, g_1, 1)])])$$
$$= \text{ (by definition of } \overline{par}_{\alpha(f_1(u_1,y_1)),(B(E))})$$
$$nf(\Rightarrow_{R_2}, \omega(g_1(y_1, z_{g_2,1}), z_{g_2,1})$$
$$[z_{g_2,1} \leftarrow nf(\Rightarrow_{R_2}, z_1[v_1 \leftarrow nf(\Rightarrow_{R_1}, f_1(u_1, y_1)[u_1 \leftarrow B(E)]),$$
$$z_1 \leftarrow z_{g_1,1}])])$$
$$= \text{ (since } nf(\Rightarrow_{R_2}, z_1[v_1, z_1 \leftarrow nf(\Rightarrow_{R_1}, f_1(u_1, y_1)[u_1 \leftarrow B(E)]), z_{g_1,1}]) = z_{g_1,1})$$
$$\omega(g_1(y_1, z_{g_1,1}), z_{g_1,1}).$$

In Example A.5 we will establish how this result corresponds to the twofold underlined subtree in Example 4.2. ◇

In order to describe the context parameters with which states of M_2 reach certain paths in a tree, we need the notion of *marking* a tree. Therefor, we define a function *mark* on arbitrary trees, which adds to every occurrence of a symbol as a superscript its path in the tree. For example, we have

$$mark(\delta(f(u, y), \delta(\alpha, \alpha))) = \delta^{\varepsilon}(f^1(u^{11}, y^{12}), \delta^2(\alpha^{21}, \alpha^{22})).$$

All operations on marked trees work in principle like on ordinary trees. Markings are ignored, e.g., in matching a tree against the left-hand side of a rewrite rule, but markings are preserved by, e.g., rewriting, substitutions or the *sub*-function. If there is, e.g., a rule $g(\delta(v_1, v_2)) \to \gamma(g(v_2))$, then this induces the rewrite step $g(\delta^{\varepsilon}(f^1(u^{11}, y^{12}), \delta^2(\alpha^{21}, \alpha^{22}))) \Rightarrow \gamma(g(\delta^2(\alpha^{21}, \alpha^{22})))$. We will use the placeholder $\bullet^{\pi}$ for an arbitrary tree that has a root symbol marked with π.

Lemma A.3 (*reachability of marked subtrees*)
For every $p \in \mathbb{N}$, $t_1, \ldots, t_p \in T_{\Sigma}$, $r \in \mathbb{N}$, $\phi \in RHS(F, \Delta, U_p, Y_r)$, $g \in G^{(s+1)}$, $g' \in G^{(s'+1)}$, $j \in \mathbb{N}_+$ and $\pi j \in paths(\phi)$: if a call of the form $g(\bullet^{\pi j}, \varrho_1, \ldots, \varrho_s)$ occurs during a reduction of $g'(mark(\phi)[u_c \leftarrow t_c, c \in [p]], z_1, \ldots, z_{s'})$ with $\Rightarrow_{R_1 \cup R_2}$, then there exist $g'' \in G^{(s''+1)}$ and $\xi, \eta_1, \ldots, \eta_{s''}$ such that

$$g'(mark(\phi)[u_c \leftarrow t_c, c \in [p]], z_1, \ldots, z_{s'}) \Rightarrow^*_{R_1 \cup R_2} \xi\,,$$

ξ contains a subtree

$$g''(sub(mark(\phi), \pi)[u_c \leftarrow t_c, c \in [p]], \eta_1, \ldots, \eta_{s''})$$

and from this call of g'' on $sub(mark(\phi), \pi)[u_c \leftarrow t_c, c \in [p]]$ results—by reduction—the $g(\bullet^{\pi j}, \varrho_1, \ldots, \varrho_s)$-call.

Proof
If a call of the form $g(\bullet^{\pi j}, \varrho_1, \ldots, \varrho_s)$ occurs during a reduction of

$$g'(mark(\phi)[u_c \leftarrow t_c, c \in [p]], z_1, \ldots, z_{s'})$$

with $\Rightarrow_{R_1 \cup R_2}$, then during this reduction a call of some $g'' \in G^{(s''+1)}$ with some context parameters $\eta_1, \ldots, \eta_{s''}$ on a tree ζ with a root symbol marked with π must

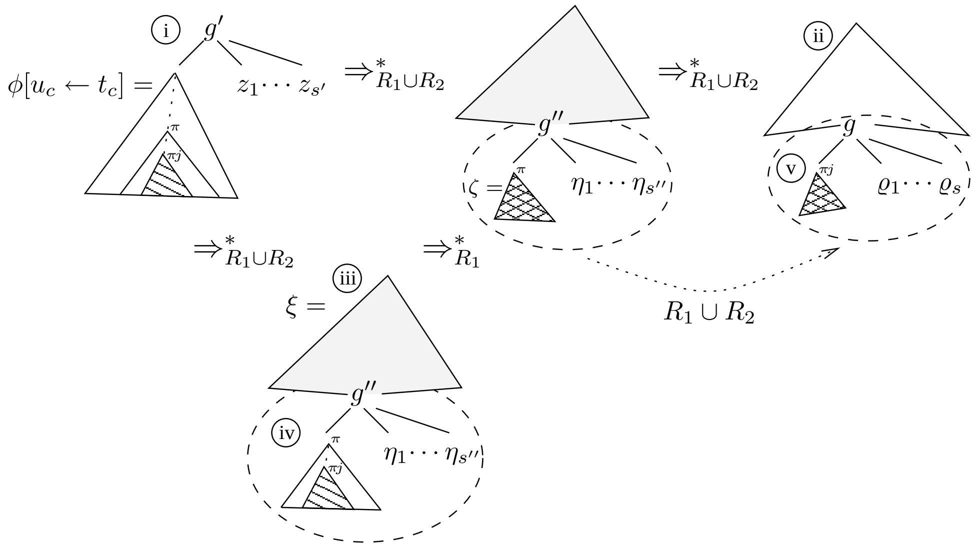

Fig. A 1. If **i** reduces to **ii**, then **iii** exists such that a reduction of **iv** contains a **v**.

have occurred beforehand, from which the $g(\bullet^{\pi j}, \varrho_1, \ldots, \varrho_s)$ results. This situation is shown in the first row of Fig. A 1.

The tree ζ can only be obtained from the subtree of $\phi[u_c \leftarrow t_c, c \in [p]]$ at path π by $\Rightarrow_{R_1}$-reduction steps. By shifting these steps to the end of the first reduction in the first row of Fig. A 1, we obtain the detour reduction via ξ also shown in the figure. Hence, the subtree $g''(sub(mark(\phi), \pi)[u_c \leftarrow t_c, c \in [p]], \eta_1, \ldots, \eta_{s''})$ of ξ reduces to $g''(\zeta, \eta_1, \ldots, \eta_{s''})$ by $\Rightarrow^*_{R_1}$ and from this results—by further reduction—the $g(\bullet^{\pi j}, \varrho_1, \ldots, \varrho_s)$-call. $\square$

We will use the following principle of proof by simultaneous induction (Engelfriet & Vogler, 1985; Kühnemann & Vogler, 1994; Fülöp & Vogler, 1998).

Proof principle
Let Σ be a ranked alphabet. Let **I** and **II** be statements, where **I** has a free variable $t \in T_\Sigma$ and **II** has free variables $p \in \mathbb{N}$ and $t_1, \ldots, t_p \in T_\Sigma$. If

I $\Leftarrow$ **II :** for every $p \in \mathbb{N}$ and $t_1, \ldots, t_p \in T_\Sigma$ such that **II** holds, we can prove for every $\sigma \in \Sigma^{(p)}$ that **I** holds for $t = \sigma(t_1, \ldots, t_p)$, and

II $\Leftarrow$ **I :** for every $p \in \mathbb{N}$ and $t_1, \ldots, t_p \in T_\Sigma$ such that **I** holds for each of the $t_1, \ldots, t_p$, we can prove that **II** holds,

then we have proven that **I** holds for every $t \in T_\Sigma$ and that **II** holds for every $p \in \mathbb{N}$ and $t_1, \ldots, t_p \in T_\Sigma$. $\diamond$

In the proofs we will frequently use "reordering of normal form computations and/or of substitutions" to justify the manipulation of expressions. Each of these intuitive steps could be proven by induction of a much lower complexity than the overall proof structure. Since these inductions nevertheless would blow up this appendix considerably, we omit them.

The following theorem establishes the key property of the functions introduced in Definition A.1.

Theorem A.4 ($\overline{par}_{f,t}$*-functions answer question* **Q**)
For every $t \in T_\Sigma$, $f \in F^{(r+1)}$, $g \in G^{(s+1)}$, $g' \in G^{(s'+1)}$, $k \in [r]$ and $l \in [s]$: if a call of the form $g(y_k, \varrho_1, \ldots, \varrho_s)$ occurs during a reduction of $g'(f(t, y_1, \ldots, y_r), z_1, \ldots, z_{s'})$ with $\Rightarrow_{R_1 \cup R_2}$, then $\overline{par}_{f,t}(k, g, l) \in RHS(G, \Omega, Y_r, \{z_{g',1}, \ldots, z_{g',s'}\})$ and $nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l) = \overline{par}_{f,t}(k, g, l)[z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}]$.

Proof
We prove the following two statements by simultaneous induction, where statement **I** is exactly the same as the statement of the theorem.

I. For every $t \in T_\Sigma$:
For every $f \in F^{(r+1)}$, $g \in G^{(s+1)}$, $g' \in G^{(s'+1)}$, $k \in [r]$ and $l \in [s]$: if a call of the form $g(y_k, \varrho_1, \ldots, \varrho_s)$ occurs during a reduction of $g'(f(t, y_1, \ldots, y_r), z_1, \ldots, z_{s'})$ with $\Rightarrow_{R_1 \cup R_2}$, then:

$$\overline{par}_{f,t}(k, g, l) \in RHS(G, \Omega, Y_r, \{z_{g',1}, \ldots, z_{g',s'}\}) \text{ and}$$

$$nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l) = \overline{par}_{f,t}(k, g, l)[z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}].$$

II. For every $p \in \mathbb{N}, t_1, \ldots, t_p \in T_\Sigma$:
For every $(r+1) \in rank(F)$ with $r > 0$, $\phi \in RHS(F, \Delta, U_p, Y_r)$, $\pi \in paths(\phi)$ such that $lab(\phi, \pi) \notin U_p$, $g \in G^{(s+1)}$, $g' \in G^{(s'+1)}$ and $l \in [s]$: if a call of the form $g(\bullet^\pi, \varrho_1, \ldots, \varrho_s)$ occurs during a reduction of $g'(mark(\phi)[u_c \leftarrow t_c, c \in [p]], z_1, \ldots, z_{s'})$ with $\Rightarrow_{R_1 \cup R_2}$, then:

$$\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g, l) \in RHS(G, \Omega, Y_r, \{z_{g',1}, \ldots, z_{g',s'}\}) \text{ and}$$

$$nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l) = \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g, l)[z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}].$$

I $\Leftarrow$ **II :** Let $t = \sigma(t_1, \ldots, t_p)$ for some $\sigma \in \Sigma^{(p)}$ and $t_1, \ldots, t_p \in T_\Sigma$.
If a call $g(y_k, \varrho_1, \ldots, \varrho_s)$ occurs during a reduction of

$$g'(f(\sigma(t_1, \ldots, t_p), y_1, \ldots, y_r), z_1, \ldots, z_{s'})$$

with $\Rightarrow_{R_1 \cup R_2}$, then it must occur during a reduction of

$$g'(rhs_{f,\sigma}[u_c \leftarrow t_c, c \in [p]], z_1, \ldots, z_{s'}).$$

This is only possible if there is a—unique, by the non-copying property of M_1—path π in $paths(rhs_{f,\sigma})$ with $lab(rhs_{f,\sigma}, \pi) = y_k$. Consequently, the $\varrho_1, \ldots, \varrho_s$ appear in an occurrence $g({y_k}^\pi, \varrho_1, \ldots, \varrho_s)$, resulting from a reduction of

$$g'(mark(rhs_{f,\sigma})[u_c \leftarrow t_c, c \in [p]], z_1, \ldots, z_{s'}).$$

By induction hypothesis **II** this means that

$$\overline{par}_{rhs_{f,\sigma},(t_1,\ldots,t_p)}(\pi, g, l) \in RHS(G, \Omega, Y_r, \{z_{g',1}, \ldots, z_{g',s'}\}) \text{ and}$$

$$nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l) = \overline{par}_{rhs_{f,\sigma},(t_1,\ldots,t_p)}(\pi, g, l)[z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}]\,,$$

which by $\overline{par}_{f,\sigma(t_1,\ldots,t_p)}(k, g, l) = \overline{par}_{rhs_{f,\sigma},(t_1,\ldots,t_p)}(\pi, g, l)$ is exactly what we had to prove.

II $\Leftarrow$ I : For fixed r and ϕ, we prove the claim by induction on the prefix-order of paths in ϕ:

- In the base case ε, we have a call of the form $g(\bullet^{\varepsilon}, \varrho_1, \ldots, \varrho_s)$ during reduction of $g'(mark(\phi)[u_c \leftarrow t_c, c \in [p]], z_1, \ldots, z_{s'})$. This is only possible if $g = g'$ and $\varrho_1, \ldots, \varrho_s = z_1, \ldots, z_{s'}$, in which case we have by Definition A.1: $\overline{par}_{\phi,(t_1,\ldots,t_p)}(\varepsilon, g, l) = z_{g,l} \in RHS(G, \Omega, Y_r, \{z_{g',1}, \ldots, z_{g',s'}\})$ and $nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l) = z_l = \overline{par}_{\phi,(t_1,\ldots,t_p)}(\varepsilon, g, l)[z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}]$.

- In the inductive case $\pi j \in paths(\phi)$—with $\pi \in paths(\phi)$, $j \in \mathbb{N}_+$ and $lab(\phi, \pi j) \notin U_p$—we have a call of the form $g(\bullet^{\pi j}, \varrho_1, \ldots, \varrho_s)$ during reduction of $g'(mark(\phi)[u_c \leftarrow t_c, c \in [p]], z_1, \ldots, z_{s'})$. By Lemma A.3 there exist $g'' \in G^{(s''+1)}$ and $\eta_1, \ldots, \eta_{s''}$ such that reducing

$$g'(mark(\phi)[u_c \leftarrow t_c, c \in [p]], z_1, \ldots, z_{s'})$$

leads to an occurrence of

$$g''(sub(mark(\phi), \pi)[u_c \leftarrow t_c, c \in [p]], \eta_1, \ldots, \eta_{s''})$$

and from this call of g'' on $sub(mark(\phi), \pi)[u_c \leftarrow t_c, c \in [p]]$ results—by reduction—a call of the form $g(\bullet^{\pi j}, \varrho_1, \ldots, \varrho_s)$.
By applying the induction hypothesis for π we know that for every $m \in [s'']$, we have

$$\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g'', m) \in RHS(G, \Omega, Y_r, \{z_{g',1}, \ldots, z_{g',s'}\}) \text{ and}$$

$$nf(\Rightarrow_{R_1 \cup R_2}, \eta_m) = \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g'', m)[z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}].$$

We proceed by case distinction on $lab(\phi, \pi)$:

$\underline{lab(\phi, \pi) = \delta}$ for some $\delta \in \Delta^{(q)}$ and $j \in [q]$:
On the one hand, we know that from reduction of the call

$$g''(\delta^{\pi}(sub(mark(\phi), \pi 1), \ldots, sub(mark(\phi), \pi q))[u_c \leftarrow t_c, c \in [p]], \eta_1, \ldots, \eta_{s''})$$

results a call of the form $g(\bullet^{\pi j}, \varrho_1, \ldots, \varrho_s)$. Consequently, there must be a rule

$$g''(\delta(v_1, \ldots, v_q), z_1, \ldots, z_{s''}) \rightarrow \ldots g(v_j, \psi_1, \ldots, \psi_s) \ldots$$

in R_2 (for some $\psi_1, \ldots, \psi_s \in RHS(G, \Omega, V_q, Z_{s''})$) and:

$$\begin{aligned} nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l) &= nf(\Rightarrow_{R_1 \cup R_2}, \psi_l[v_d \leftarrow sub(\phi, \pi d)[u_c \leftarrow t_c, c \in [p]], d \in [q], \\ &\qquad z_m \leftarrow \eta_m, m \in [s'']]) \\ &= \text{(by reordering of normal form computations)} \\ & nf(\Rightarrow_{R_2}, \psi_l[v_d \leftarrow nf(\Rightarrow_{R_1}, sub(\phi, \pi d)[u_c \leftarrow t_c, c \in [p]]), d \in [q], \\ &\qquad z_m \leftarrow nf(\Rightarrow_{R_1 \cup R_2}, \eta_m), m \in [s'']]) \\ &= \text{(by the above induction hypothesis for } \pi) \end{aligned}$$

$$nf(\Rightarrow_{R_2}, \psi_l[v_d \ \leftarrow nf(\Rightarrow_{R_1}, sub(\phi, \pi d)[u_c \leftarrow t_c, c \in [p]]), d \in [q], z_m \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g'', m) [z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}], m \in [s'']])$$

= (since the only occurrences of $z_{g',1}, \ldots, z_{g',s'}$ are in the expressions substituted for the z_m)

$$nf(\Rightarrow_{R_2}, \psi_l[v_d \ \leftarrow nf(\Rightarrow_{R_1}, sub(\phi, \pi d)[u_c \leftarrow t_c, c \in [p]]), d \in [q], z_m \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g'', m), m \in [s'']]) [z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}].$$

On the other hand, taking into account that the call $g(v_j, \psi_1, \ldots, \psi_s)$ in the above rule from R_2 must be the unique occurrence of a $g(v_j, \ldots)$-call in the δ-rules of the weakly single-use mtt M_2, we have by definition of $\overline{par}_{\phi,(t_1,\ldots,t_p)}$:

$$\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi j, g, l) = nf(\Rightarrow_{R_2}, \psi_l[v_d \ \leftarrow nf(\Rightarrow_{R_1}, sub(\phi, \pi d)[u_c \leftarrow t_c, c \in [p]]), d \in [q], z_m \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g'', m), m \in [s'']]).$$

Since for every $m \in [s'']$ we know by the induction hypothesis for π that $\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g'', m) \in RHS(G, \Omega, Y_r, \{z_{g',1}, \ldots, z_{g',s'}\})$, and since for every $d \in [q]$ we have $nf(\Rightarrow_{R_1}, sub(\phi, \pi d)[u_c \leftarrow t_c, c \in [p]]) \in T_\Delta(Y_r)$ (because $\phi \in RHS(F, \Delta, U_p, Y_r)$), this implies:

$$\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi j, g, l) \in RHS(G, \Omega, Y_r, \{z_{g',1}, \ldots, z_{g',s'}\}).$$

We also have that the previously computed equivalent of $nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l)$ equals

$$\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi j, g, l)[z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}]\,,$$

which proves the claim.

$\underline{lab(\phi, \pi) = f}$ for some $f \in F^{(q+1)}$, $1 \leq j-1 \leq q$ and $lab(\phi, \pi 1) = u_i \in U_p$: On the one hand, we know that from reduction of the call

$$g''(f^\pi({u_i}^{\pi 1}, sub(mark(\phi), \pi 2), \ldots, sub(mark(\phi), \pi(q+1)))[u_c \leftarrow t_c, c \in [p]], \eta_1, \ldots, \eta_{s''})$$

results a call of the form $g(\bullet^{\pi j}, \varrho_1, \ldots, \varrho_s)$. This means that a call of g reaches the $(j-1)$st context parameter position of the call of f^π on t_i. By the induction hypothesis **I** for t_i we know that for a call of the form $g(y_{j-1}, \varrho'_1, \ldots, \varrho'_s)$ resulting from a reduction of

$$g''(f(t_i, y_1, \ldots, y_q), z_1, \ldots, z_{s''})\,,$$

we have:

$$\overline{par}_{f,t_i}(j-1, g, l) \in RHS(G, \Omega, Y_q, \{z_{g'',1}, \ldots, z_{g'',s''}\}) \text{ and}$$

$$nf(\Rightarrow_{R_1 \cup R_2}, \varrho'_l) = \overline{par}_{f,t_i}(j-1, g, l)[z_{g'',1}, \ldots, z_{g'',s''} \leftarrow z_1, \ldots, z_{s''}].$$

For our call $g(\bullet^{\pi j}, \varrho_1, \ldots, \varrho_s)$ resulting from reducing

$$\begin{aligned} g''(f^\pi(t_i, &sub(mark(\phi), \pi 2)[u_c \leftarrow t_c, c \in [p]], \\ &\ldots, \\ &sub(mark(\phi), \pi(q+1))[u_c \leftarrow t_c, c \in [p]]), \\ \eta_1, \ldots, \eta_{s''}) \,, \end{aligned}$$

this implies the following calculation:

$$\begin{aligned}
& nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l) = nf(\Rightarrow_{R_1 \cup R_2}, \varrho'_l[y_b \leftarrow sub(\phi, \pi(b+1))[u_c \leftarrow t_c, c \in [p]], b \in [q], \\
& \qquad z_m \leftarrow \eta_m, m \in [s'']]) \\
& = \text{(by reordering of normal form computation)} \\
& nf(\Rightarrow_{R_1 \cup R_2}, nf(\Rightarrow_{R_1 \cup R_2}, \varrho'_l)[y_b \leftarrow sub(\phi, \pi(b+1))[u_c \leftarrow t_c, c \in [p]], b \in [q], \\
& \qquad z_m \leftarrow \eta_m, m \in [s'']]) \\
& = \text{(by } nf(\Rightarrow_{R_1 \cup R_2}, \varrho'_l) = \overline{par}_{f,t_i}(j-1, g, l)[z_{g'',1}, \ldots, z_{g'',s''} \leftarrow z_1, \ldots, z_{s''}]) \\
& nf(\Rightarrow_{R_1 \cup R_2}, \overline{par}_{f,t_i}(j-1, g, l)[z_{g'',1}, \ldots, z_{g'',s''} \leftarrow z_1, \ldots, z_{s''}] \\
& \qquad [y_b \leftarrow sub(\phi, \pi(b+1))[u_c \leftarrow t_c, c \in [p]], b \in [q], \\
& \qquad z_m \leftarrow \eta_m, m \in [s'']]) \\
& = \text{(by reordering of normal form computations)} \\
& nf(\Rightarrow_{R_2}, \overline{par}_{f,t_i}(j-1, g, l)[z_{g'',1}, \ldots, z_{g'',s''} \leftarrow z_1, \ldots, z_{s''}] \\
& \qquad [y_b \leftarrow nf(\Rightarrow_{R_1}, sub(\phi, \pi(b+1))[u_c \leftarrow t_c, c \in [p]]), b \in [q], \\
& \qquad z_m \leftarrow nf(\Rightarrow_{R_1 \cup R_2}, \eta_m), m \in [s'']]) \\
& = \text{(by the above induction hypothesis for } \pi) \\
& nf(\Rightarrow_{R_2}, \overline{par}_{f,t_i}(j-1, g, l)[z_{g'',1}, \ldots, z_{g'',s''} \leftarrow z_1, \ldots, z_{s''}] \\
& \qquad [y_b \leftarrow nf(\Rightarrow_{R_1}, sub(\phi, \pi(b+1))[u_c \leftarrow t_c, c \in [p]]), b \in [q], \\
& \qquad z_m \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g'', m) \\
& \qquad\qquad [z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}], m \in [s'']]) \\
& = \text{(by composing substitutions)} \\
& nf(\Rightarrow_{R_2}, \overline{par}_{f,t_i}(j-1, g, l)[y_b \leftarrow nf(\Rightarrow_{R_1}, sub(\phi, \pi(b+1)) \\
& \qquad\qquad [u_c \leftarrow t_c, c \in [p]]), b \in [q], \\
& \qquad z_{g'',m} \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g'', m) \\
& \qquad\qquad [z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}], \\
& \qquad m \in [s'']]).
\end{aligned}$$

Since we reasoned above that

$$\overline{par}_{f,t_i}(j-1, g, l) \in RHS(G, \Omega, Y_q, \{z_{g'',1}, \ldots, z_{g'',s''}\}) \,,$$

$\overline{par}_{f,t_i}(j-1, g, l)$ does not contain variables from $Z_G \backslash \{z_{g'',1}, \ldots, z_{g'',s''}\}$. Hence, we can transform the previous expression into the following[1]:

[1] It does not matter which values we replace for the variables from $Z_G \backslash \{z_{g'',1}, \ldots, z_{g'',s''}\}$, because they anyway do not occur in $\overline{par}_{f,t_i}(j-1, g, l)$. We only need to ensure that for the $z_{g'',1}, \ldots, z_{g'',s''}$ the same values are substituted as in the previous expression.

$$
\begin{aligned}
&nf(\Rightarrow_{R_2}, \overline{par}_{f,t_i}(j-1,g,l)[y_b \quad \leftarrow nf(\Rightarrow_{R_1}, sub(\phi,\pi(b+1)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad [u_c \leftarrow t_c, c \in [p]]), b \in [q], \\
&\qquad\qquad\qquad z_{g''',m} \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g''', m) \\
&\qquad\qquad\qquad\qquad [z_{g',1},\ldots,z_{g',s'} \leftarrow z_1,\ldots,z_{s'}], \\
&\qquad\qquad\qquad g''' \in G^{(s'''+1)}, m \in [s''']])
\end{aligned}
$$

= (by composition of substitutions, using that the normal forms replaced for the $y_1,\ldots,y_q$ do not contain $z_{g',1},\ldots,z_{g',s'}$)

$$
\begin{aligned}
&nf(\Rightarrow_{R_2}, \overline{par}_{f,t_i}(j-1,g,l) \\
&\qquad [y_b \quad \leftarrow nf(\Rightarrow_{R_1}, sub(\phi,\pi(b+1))[u_c \leftarrow t_c, c \in [p]]), b \in [q], \\
&\qquad z_{g''',m} \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g''', m),\ g''' \in G^{(s'''+1)}, m \in [s''']]) \\
&\quad [z_{g',1},\ldots,z_{g',s'} \leftarrow z_1,\ldots,z_{s'}].
\end{aligned}
$$

On the other hand, by the definition of $\overline{par}_{\phi,(t_1,\ldots,t_p)}$, we have:

$$
\begin{aligned}
&\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi j, g, l) \\
&= nf(\Rightarrow_{R_2}, \overline{par}_{f,t_i}(j-1,g,l) \\
&\qquad [y_b \quad \leftarrow nf(\Rightarrow_{R_1}, sub(\phi,\pi(b+1))[u_c \leftarrow t_c, c \in [p]]), b \in [q], \\
&\qquad z_{g''',m} \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g''', m),\ g''' \in G^{(s'''+1)}, m \in [s''']]).
\end{aligned}
$$

Since for every $m \in [s'']$ we know by the induction hypothesis for π that $\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g'', m) \in RHS(G,\Omega,Y_r,\{z_{g',1},\ldots,z_{g',s'}\})$, and since for every $b \in [q]$ we have $nf(\Rightarrow_{R_1}, sub(\phi,\pi(b+1))[u_c \leftarrow t_c, c \in [p]]) \in T_\Delta(Y_r)$, this together with $\overline{par}_{f,t_i}(j-1,g,l) \in RHS(G,\Omega,Y_q,\{z_{g'',1},\ldots,z_{g'',s''}\})$ implies:

$$\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi j, g, l) \in RHS(G,\Omega,Y_r,\{z_{g',1},\ldots,z_{g',s'}\}).$$

We also have that the previously computed equivalent of $nf(\Rightarrow_{R_1\cup R_2}, \varrho_l)$ equals

$$\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi j, g, l)[z_{g',1},\ldots,z_{g',s'} \leftarrow z_1,\ldots,z_{s'}]\,,$$

which proves the claim. □

In Subsection 4.5 we posed the following question:

Q : "Given two states $f \in F^{(r+1)}$ and $g \in G^{(s+1)}$ and some input tree t for M_1, can we for every state g' of M_2 and every context variable y_k from $y_1,\ldots,y_r$ uniquely determine, what will be the context parameters in every occurrence of a call of g' on y_k during the reduction of $g(f(t,y_1,\ldots,y_r),\eta_1,\ldots,\eta_s)$?"

Because of Theorem A.4, we can now answer this question positively by stating that the lth context parameter of every such call of g' on y_k will be equal—modulo further reductions—to $\overline{par}_{f,t}(k,g',l)[z_{g,1},\ldots,z_{g,s} \leftarrow \eta_1,\ldots,\eta_s]$.

Example A.5 (*$\overline{par}_{f,t}$-functions answer question* **Q**)
For the mtts M_1 and M_2 from Example 4.2 we computed in Example A.2 the following:

$$\overline{par}_{f_1,A(B(E))}(1,g_2,1) = \omega(g_1(y_1,z_{g_1,1}),z_{g_1,1}) \in RHS(G,\Omega,Y_1,\{z_{g_1,1}\}).$$

Theorem A.4 now says that if a call of the form $g_2(y_1, \varrho_1)$ occurs during a reduction of $g_1(f_1(A(B(E)), y_1), z_1)$ with $\Rightarrow_{R_1 \cup R_2}$, then

$$nf(\Rightarrow_{R_1 \cup R_2}, \varrho_1) = \omega(g_1(y_1, z_{g_1,1}), z_{g_1,1})[z_{g_1,1} \leftarrow z_1]$$

must hold. In fact, we have in Example 4.2 seen the following reduction:

$$g_1(f_1(A(B(E)), y_1), z_1) \Rightarrow^*_{R_1 \cup R_2} g_1(\beta(y_1), \underline{g_2(y_1, \underline{\omega(g_1(y_1, z_1), z_1)})}). \quad \diamond$$

Next, we define—for every state f and every input tree t of M_1—two auxiliary notions that will be needed in the further proofs. The first is a unary predicate on pairs, consisting of a context parameter position k of f and some state g of M_2. It reflects whether a call of g on the kth context parameter of f can result from processing the intermediate result generated by M_1 (with f on t) with states of M_2. The second auxiliary notion is a binary relation intended to represent the possible nesting of occurrences of such calls.

Definition A.6 (unary $\vdash_{f,t}$ *and binary* $\prec_{f,t}$*)*
For every $f \in F^{(r+1)}$, $t \in T_\Sigma$, $k, k' \in [r]$ and $g, g' \in G$, we write:

1. $\vdash_{f,t} (k, g)$ iff there exists a $g''' \in G^{(s'''+1)}$ such that there exists a reduction of $g'''(f(t, y_1, \ldots, y_r), z_1, \ldots, z_{s'''})$ with $\Rightarrow_{R_1 \cup R_2}$ in which a call of the form $g(y_k, \ldots)$ occurs.
2. $(k, g) \prec_{f,t} (k', g')$ iff there exists an $l \in [rank_G(g) - 1]$ such that $\overline{par}_{f,t}(k, g, l)$ contains an occurrence of a $g'(y_{k'}, \ldots)$-call.

For every subset $\mathcal{C} \subseteq [r] \times G$, we write $(k, g) \prec^*_{f,t} \mathcal{C}$ iff there exists a $(k''', g''') \in \mathcal{C}$ with $(k, g) \prec^*_{f,t} (k''', g''')$. Analogous for $\prec^+_{f,t}$. $\diamond$

In Subsection 4.10 we claimed that for non-copying M_1 and weakly single-use M_2, we cannot reach a situation where two calls corresponding to one and the same such pair are nested, which would be a similar situation as in "counterexample" (b) in Example 4.3. Now, we can formalise this statement (in Lemma A.8), after having established two properties of the notions introduced in Definition A.6.

Lemma A.7 (propagation of $\vdash_{f,t}$ *and* $\prec_{f,t}$*)*
For every $f \in F^{(r+1)}$, $t \in T_\Sigma$, $k, k', k'' \in [r]$ and $g, g', g'' \in G$:

1. If $\vdash_{f,t} (k, g)$ and $(k, g) \prec_{f,t} (k', g')$, then $\vdash_{f,t} (k', g')$.
2. If $\vdash_{f,t} (k, g)$ and $(k, g) \prec_{f,t} (k', g') \prec_{f,t} (k'', g'')$, then $(k, g) \prec_{f,t} (k'', g'')$.

Proof
Let $g \in G^{(s+1)}$ and $g' \in G^{(s'+1)}$.

1. If $\vdash_{f,t} (k, g)$ and $(k, g) \prec_{f,t} (k', g')$, then we know by Definition A.6 that there exists a $g''' \in G^{(s'''+1)}$ such that a call of the form $g(y_k, \varrho_1, \ldots, \varrho_s)$ occurs during a reduction of $g'''(f(t, y_1, \ldots, y_r), z_1, \ldots, z_{s'''})$ with $\Rightarrow_{R_1 \cup R_2}$, and that there exists an $l \in [s]$ such that $\overline{par}_{f,t}(k, g, l)$ contains an occurrence of a $g'(y_{k'}, \ldots)$-call. It follows from Theorem A.4 that $nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l)$ is equal to $\overline{par}_{f,t}(k, g, l)[z_{g''',1}, \ldots, z_{g''',s'''} \leftarrow z_1, \ldots, z_{s'''}]$ and hence contains a call of the form $g'(y_{k'}, \varrho'_1, \ldots, \varrho'_{s'})$. Since this call occurs in a $\Rightarrow_{R_1 \cup R_2}$-reduction of

$g'''(f(t, y_1, \ldots, y_r), z_1, \ldots, z_{s'''})$, it follows that $\vdash_{f,t} (k', g')$.

2. If additionally $(k', g') \prec_{f,t} (k'', g'')$, then by Definition A.6(2) there exists an $l' \in [s']$ such that $\overline{par}_{f,t}(k', g', l')$ contains an occurrence of a $g''(y_{k''}, \ldots)$-call. Again by Theorem A.4 follows that $nf(\Rightarrow_{R_1 \cup R_2}, \varrho'_{l'})$ contains a call of the form $g''(y_{k''}, \ldots)$. Since—as a subtree of $nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l)$—$\varrho'_{l'}$ is a $\Rightarrow_{R_1 \cup R_2}$-normal form, this $g''(y_{k''}, \ldots)$-call is in fact a subtree of $\varrho'_{l'}$ and hence of $nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l) = \overline{par}_{f,t}(k, g, l)[z_{g''',1}, \ldots, z_{g''',s'''} \leftarrow z_1, \ldots, z_{s'''}]$. According to Definition A.6(2) it follows that $(k, g) \prec_{f,t} (k'', g'')$. □

Lemma A.8 (*no cycles in* $\prec_{f,t}$)
For every $f \in F^{(r+1)}$, $t \in T_\Sigma$, $k \in [r]$, $g \in G$: if $\vdash_{f,t} (k, g)$, then not $(k, g) \prec^+_{f,t} (k, g)$.

Proof
Let $g \in G^{(s+1)}$, and assume $\vdash_{f,t} (k, g)$ and $(k, g) \prec^+_{f,t} (k, g)$. By repeatedly applying Lemma A.7(2), we obtain $(k, g) \prec_{f,t} (k, g)$. By Definition A.6(2) this means that there must exist an $l \in [s]$ such that $\overline{par}_{f,t}(k, g, l)$ contains an occurrence of a $g(y_k, \ldots)$-call. Since $\vdash_{f,t} (k, g)$, there must exist a $g' \in G^{(s'+1)}$ such that during a reduction of $g'(f(t, y_1, \ldots, y_r), z_1, \ldots, z_{s'})$ with $\Rightarrow_{R_1 \cup R_2}$ a call of the form $g(y_k, \varrho_1, \ldots, \varrho_s)$ occurs. By Theorem A.4 we have:

$$nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l) = \overline{par}_{f,t}(k, g, l)[z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}]\,,$$

which contains a $g(y_k, \varrho'_1, \ldots, \varrho'_s)$-call for some $\varrho'_1, \ldots, \varrho'_s \in RHS(G, \Omega, Y_r, Z_{s'})$. Since this call does occur in a $\Rightarrow_{R_1 \cup R_2}$-reduction of $g'(f(t, y_1, \ldots, y_r), z_1, \ldots, z_{s'})$, we have again by Theorem A.4:

$$nf(\Rightarrow_{R_1 \cup R_2}, \varrho'_l) = \overline{par}_{f,t}(k, g, l)[z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}].$$

Since—as a subtree of $nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l)$—$\varrho'_l$ is a $\Rightarrow_{R_1 \cup R_2}$-normal form, this implies:

$$\varrho'_l = \overline{par}_{f,t}(k, g, l)[z_{g',1}, \ldots, z_{g',s'} \leftarrow z_1, \ldots, z_{s'}] = nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l).$$

But ϱ'_l was shown above to be a proper subtree of $nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l)$, which leads to a contradiction. □

We make two simple observations about sufficient conditions for the notions introduced in Definition A.6.

Lemma A.9 (*sufficient conditions for* $\vdash_{f,t}$ *and* $\prec_{f,t}$)
For every $f \in F^{(r+1)}$, $t \in T_\Sigma$, $k, k' \in [r]$, $g \in G^{(s+1)}$ and $g' \in G$:

1. If $y_{k,g'}$ occurs in $nf(\Rightarrow_{Pre}, nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(t, y_1, \ldots, y_r)), z_1, \ldots, z_s)))$, then $\vdash_{f,t} (k, g')$.
2. If for some $l \in [s]$, the normal form $nf(\Rightarrow_{Pre}, \overline{par}_{f,t}(k, g, l))$ contains $y_{k',g'}$, then $(k, g) \prec_{f,t} (k', g')$.

Proof
Follows easily from the definition of *Pre* and Definition A.6. □

Now, we prove an important technical lemma about the correctness of precomputing translations of the first mtt's context parameters with states of M_2 (given the correct context parameters of these states on reaching M_1's context parameters), and then providing these translations in the appropriate positions of the result obtained by computing in ignorance of M_1's concrete context parameter values.

Lemma A.10 (*precomputing translations*)
For every $f \in F^{(q+1)}$, $g \in G^{(s+1)}$, $t \in T_\Sigma$, $r \in \mathbb{N}$ and $\theta_1, \ldots, \theta_q \in T_\Delta(Y_r)$:

$$\begin{aligned} & nf(\Rightarrow_{Pre}, nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(t, y_1, \ldots, y_q)), z_1, \ldots, z_s))) \\ & \quad [y_{k,g'} \leftarrow nf(\Rightarrow_{R_2}, g'(y_k, \overline{par}_{f,t}(k, g', 1), \ldots, \overline{par}_{f,t}(k, g', s')) \\ & \qquad\qquad [y_1, \ldots, y_q \leftarrow \theta_1, \ldots, \theta_q]) \\ & \qquad\qquad [z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s][z_{g_1,1}, \ldots, z_{g_\mu,s_\mu} \leftarrow nil, \ldots, nil], \\ & \qquad k \in [q], g' \in G^{(s'+1)}] \\ & = nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(t, \theta_1, \ldots, \theta_q)), z_1, \ldots, z_s)). \end{aligned}$$

Proof

- Firstly, consider the normal form $nf(\Rightarrow_{R_1 \cup R_2}, g(f(t, y_1, \ldots, y_q), z_1, \ldots, z_s))$. We are going to prove by structural induction that for every subtree $\varrho \in RHS(G, \Omega, Y_q, Z_s)$ of this normal form the following holds:

$$\begin{aligned} & nf(\Rightarrow_{Pre}, \varrho) \\ & \quad [y_{k,g'} \leftarrow nf(\Rightarrow_{R_2}, g'(y_k, \overline{par}_{f,t}(k, g', 1), \ldots, \overline{par}_{f,t}(k, g', s')) \\ & \qquad\qquad [y_1, \ldots, y_q \leftarrow \theta_1, \ldots, \theta_q]) \\ & \qquad\qquad [z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s][z_{g_1,1}, \ldots, z_{g_\mu,s_\mu} \leftarrow nil, \ldots, nil], \\ & \qquad k \in [q], g' \in G^{(s'+1)}] \\ & = nf(\Rightarrow_{R_2}, \varrho[y_1, \ldots, y_q \leftarrow \theta_1, \ldots, \theta_q]). \end{aligned}$$

  The cases $\varrho \in Z_s$ and $\varrho = \omega(\varrho_1, \ldots, \varrho_p)$ (for $\omega \in \Omega^{(p)}$ and $\varrho_1, \ldots, \varrho_p \in RHS(G, \Omega, Y_q, Z_s)$) are straightforward and thus omitted here.
  In the case $\varrho = g''(y_{k'}, \varrho_1, \ldots, \varrho_{s''})$ for some $y_{k'} \in Y_q$, $g'' \in G^{(s''+1)}$ and $\varrho_1, \ldots, \varrho_{s''} \in RHS(G, \Omega, Y_q, Z_s)$ we reason as follows. Since $g''(y_{k'}, \varrho_1, \ldots, \varrho_{s''})$ is a subtree of $nf(\Rightarrow_{R_1 \cup R_2}, g(f(t, y_1, \ldots, y_q), z_1, \ldots, z_s))$, we can apply Theorem A.4 to obtain that for every $l \in [s'']$ we have:

$$\overline{par}_{f,t}(k', g'', l) \in RHS(G, \Omega, Y_q, \{z_{g,1}, \ldots, z_{g,s}\}) \text{ and}$$

$$nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l) = \overline{par}_{f,t}(k', g'', l)[z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s].$$

  Hence, we calculate for every $l \in [s'']$:

$$\begin{aligned} & \overline{par}_{f,t}(k', g'', l)[y_1, \ldots, y_q \leftarrow \theta_1, \ldots, \theta_q][z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s] \\ & \qquad [z_{g_1,1}, \ldots, z_{g_\mu,s_\mu} \leftarrow nil, \ldots, nil] \\ & \quad = \text{(since the } \theta_1, \ldots, \theta_q \text{ do not contain any variables from } Z_G, \\ & \qquad \text{and } \overline{par}_{f,t}(k', g'', l) \in RHS(G, \Omega, Y_q, \{z_{g,1}, \ldots, z_{g,s}\}) \text{ contains no} \\ & \qquad \text{variables from } Z_G \backslash \{z_{g,1}, \ldots, z_{g,s}\}) \\ & \overline{par}_{f,t}(k', g'', l)[z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s][y_1, \ldots, y_q \leftarrow \theta_1, \ldots, \theta_q] \\ & \quad = \text{(since, as subtree of a normal form, } \varrho_l \text{ is itself a normal form,} \\ & \qquad \text{i.e. } \varrho_l = nf(\Rightarrow_{R_1 \cup R_2}, \varrho_l) = \overline{par}_{f,t}(k', g'', l)[z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s]) \end{aligned}$$

$\varrho_l[y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q]$.

Now, we can calculate:

$$
\begin{aligned}
& nf(\Rightarrow_{Pre}, g''(y_{k'},\varrho_1,\ldots,\varrho_{s''}))\\
&\quad [y_{k,g'} \leftarrow nf(\Rightarrow_{R_2}, g'(y_k,\overline{par}_{f,t}(k,g',1),\ldots,\overline{par}_{f,t}(k,g',s'))\\
&\qquad\qquad\qquad\qquad [y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q])\\
&\qquad\qquad\quad [z_{g,1},\ldots,z_{g,s} \leftarrow z_1,\ldots,z_s][z_{g_1,1},\ldots,z_{g_\mu,s_\mu} \leftarrow nil,\ldots,nil],\\
&\qquad\quad k\in[q], g'\in G^{(s'+1)}]\\
&\quad = \text{ (by } nf(\Rightarrow_{Pre}, g''(y_{k'},\varrho_1,\ldots,\varrho_{s''})) = y_{k',g''})\\
& nf(\Rightarrow_{R_2}, g''(y_{k'},\overline{par}_{f,t}(k',g'',1),\ldots,\overline{par}_{f,t}(k',g'',s''))\\
&\qquad\quad [y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q])\\
&\quad [z_{g,1},\ldots,z_{g,s} \leftarrow z_1,\ldots,z_s][z_{g_1,1},\ldots,z_{g_\mu,s_\mu} \leftarrow nil,\ldots,nil]\\
&\quad = \text{ (by reordering of normal form computation and substitution)}\\
& nf(\Rightarrow_{R_2}, g''(\theta_{k'},\overline{par}_{f,t}(k',g'',1)[y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q]\\
&\qquad\qquad\qquad [z_{g,1},\ldots,z_{g,s} \leftarrow z_1,\ldots,z_s]\\
&\qquad\qquad\qquad [z_{g_1,1},\ldots,z_{g_\mu,s_\mu} \leftarrow nil,\ldots,nil],\\
&\qquad\qquad\quad \ldots,\\
&\qquad\qquad \overline{par}_{f,t}(k',g'',s'')[y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q]\\
&\qquad\qquad\qquad [z_{g,1},\ldots,z_{g,s} \leftarrow z_1,\ldots,z_s]\\
&\qquad\qquad\qquad [z_{g_1,1},\ldots,z_{g_\mu,s_\mu} \leftarrow nil,\ldots,nil]))\\
&\quad = \text{ (by the equivalence shown above for every } \varrho_l[y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q])\\
& nf(\Rightarrow_{R_2}, g''(\theta_{k'},\varrho_1[y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q],\\
&\qquad\qquad\quad \ldots,\\
&\qquad\qquad \varrho_{s''}[y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q]))\\
&\quad = \text{ (by substitution)}\\
& nf(\Rightarrow_{R_2}, g''(y_{k'},\varrho_1,\ldots,\varrho_{s''})[y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q]).
\end{aligned}
$$

- By reordering of normal form computations we know that:

$$
\begin{aligned}
& nf(\Rightarrow_{R_1\cup R_2}, g(f(t,y_1,\ldots,y_q),z_1,\ldots,z_s))\\
& = nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(t,y_1,\ldots,y_q)),z_1,\ldots,z_s)).
\end{aligned}
$$

The lemma is hence proven by the following calculation:

$$
\begin{aligned}
& nf(\Rightarrow_{Pre}, nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(t,y_1,\ldots,y_q)),z_1,\ldots,z_s)))\\
&\quad [y_{k,g'} \leftarrow nf(\Rightarrow_{R_2}, g'(y_k,\overline{par}_{f,t}(k,g',1),\ldots,\overline{par}_{f,t}(k,g',s'))\\
&\qquad\qquad\qquad\qquad [y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q])\\
&\qquad\qquad\quad [z_{g,1},\ldots,z_{g,s} \leftarrow z_1,\ldots,z_s][z_{g_1,1},\ldots,z_{g_\mu,s_\mu} \leftarrow nil,\ldots,nil],\\
&\qquad\quad k\in[q], g'\in G^{(s'+1)}]\\
&\quad = \text{ (by the proposition proven in the first item}\\
&\qquad\quad \text{for } \varrho = nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(t,y_1,\ldots,y_q)),z_1,\ldots,z_s)))\\
& nf(\Rightarrow_{R_2}, nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(t,y_1,\ldots,y_q)),z_1,\ldots,z_s))\\
&\qquad\quad [y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q])\\
&\quad = \text{ (by reordering of normal form computation)}\\
& nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(t,\theta_1,\ldots,\theta_q)),z_1,\ldots,z_s)). \quad \square
\end{aligned}
$$

The following lemma states a similar property as the previous one, but for results of $\overline{par}_{f,t}$-functions.

Lemma A.11 (*purely technical*)
For every $f \in F^{(q+1)}$, $g \in G^{(s+1)}$, $t \in T_\Sigma$, $k \in [q]$, $l \in [s]$, $r \in \mathbb{N}$, $\theta_1, \ldots, \theta_q \in T_\Delta(Y_r)$, if $\vdash_{f,t} (k, g)$, then:

$$\begin{aligned}
& nf(\Rightarrow_{Pre}, \overline{par}_{f,t}(k,g,l)) \\
& \quad [y_{k',g'} \leftarrow nf(\Rightarrow_{R_2}, g'(y_{k'}, \overline{par}_{f,t}(k',g',1), \ldots, \overline{par}_{f,t}(k',g',s')) \\
& \qquad\qquad\qquad\qquad\qquad [y_1, \ldots, y_q \leftarrow \theta_1, \ldots, \theta_q]), \\
& \qquad\qquad k' \in [q], g' \in G^{(s'+1)}] \\
& = nf(\Rightarrow_{R_2}, \overline{par}_{f,t}(k,g,l)[y_1, \ldots, y_q \leftarrow \theta_1, \ldots, \theta_q]).
\end{aligned}$$

Proof
If $\vdash_{f,t} (k, g)$, then we know that there exists $g'' \in G^{(s''+1)}$ such that during a reduction of $g''(f(t, y_1, \ldots, y_q), z_1, \ldots, z_{s''})$ with $\Rightarrow_{R_1 \cup R_2}$ a call of the form $g(y_k, \eta_1, \ldots, \eta_s)$ occurs. By Theorem A.4,

$$\overline{par}_{f,t}(k,g,l) \in RHS(G, \Omega, Y_q, \{z_{g'',1}, \ldots, z_{g'',s''}\}) \text{ and}$$

$$nf(\Rightarrow_{R_1 \cup R_2}, \eta_l) = \overline{par}_{f,t}(k,g,l)[z_{g'',1}, \ldots, z_{g'',s''} \leftarrow z_1, \ldots, z_{s''}].$$

Now, we prove by structural induction on the subtrees of $\overline{par}_{f,t}(k,g,l)$ that for every such subtree $\varrho \in RHS(G, \Omega, Y_q, \{z_{g'',1}, \ldots, z_{g'',s''}\})$ holds:

$$\begin{aligned}
& nf(\Rightarrow_{Pre}, \varrho) \\
& \quad [y_{k',g'} \leftarrow nf(\Rightarrow_{R_2}, g'(y_{k'}, \overline{par}_{f,t}(k',g',1), \ldots, \overline{par}_{f,t}(k',g',s')) \\
& \qquad\qquad\qquad\qquad\qquad [y_1, \ldots, y_q \leftarrow \theta_1, \ldots, \theta_q]), \\
& \qquad\qquad k' \in [q], g' \in G^{(s'+1)}] \\
& = nf(\Rightarrow_{R_2}, \varrho[y_1, \ldots, y_q \leftarrow \theta_1, \ldots, \theta_q]).
\end{aligned}$$

The cases $\varrho \in \{z_{g'',1}, \ldots, z_{g'',s''}\}$ and $\varrho = \omega(\varrho_1, \ldots, \varrho_p)$ for $\omega \in \Omega^{(p)}$ and $\varrho_1, \ldots, \varrho_p \in RHS(G, \Omega, Y_q, \{z_{g'',1}, \ldots, z_{g'',s''}\})$ are straightforward and thus omitted here.

In the case $\varrho = g'''(y_{k'''}, \varrho_1, \ldots, \varrho_{s'''})$ for some $y_{k'''} \in Y_q$, $g''' \in G^{(s'''+1)}$ and $\varrho_1, \ldots, \varrho_{s'''} \in RHS(G, \Omega, Y_q, \{z_{g'',1}, \ldots, z_{g'',s''}\})$ we reason as follows. Since

$$g'''(y_{k'''}, \varrho_1, \ldots, \varrho_{s'''})[z_{g'',1}, \ldots, z_{g'',s''} \leftarrow z_1, \ldots, z_{s''}]$$

is a subtree of $nf(\Rightarrow_{R_1 \cup R_2}, \eta_l)$ that occurs during a $\Rightarrow_{R_1 \cup R_2}$-reduction of

$$g''(f(t, y_1, \ldots, y_q), z_1, \ldots, z_{s''}),$$

we obtain from Theorem A.4 that for every $l''' \in [s''']$ we have:

$$\overline{par}_{f,t}(k''', g''', l''') \in RHS(G, \Omega, Y_q, \{z_{g'',1}, \ldots, z_{g'',s''}\}) \text{ and}$$

$$\begin{aligned}
& nf(\Rightarrow_{R_1 \cup R_2}, \varrho_{l'''}[z_{g'',1}, \ldots, z_{g'',s''} \leftarrow z_1, \ldots, z_{s''}]) \\
& = \overline{par}_{f,t}(k''', g''', l''')[z_{g'',1}, \ldots, z_{g'',s''} \leftarrow z_1, \ldots, z_{s''}].
\end{aligned}$$

Since, as a subtree of $nf(\Rightarrow_{R_1 \cup R_2}, \eta_l)$, also $\varrho_{l'''}[z_{g'',1}, \ldots, z_{g'',s''} \leftarrow z_1, \ldots, z_{s''}]$ is a normal form, we have

$$\begin{aligned}
& \varrho_{l'''}[z_{g'',1}, \ldots, z_{g'',s''} \leftarrow z_1, \ldots, z_{s''}] \\
& = \overline{par}_{f,t}(k''', g''', l''')[z_{g'',1}, \ldots, z_{g'',s''} \leftarrow z_1, \ldots, z_{s''}]
\end{aligned}$$

and consequently (since $z_1,\ldots,z_{s''}$ do neither occur in $\varrho_{l'''}$ nor in $\overline{par}_{f,t}(k''',g''',l''')$) for every $l''' \in [s''']$: $\varrho_{l'''} = \overline{par}_{f,t}(k''',g''',l''')$. Hence, we can calculate:

$$\begin{aligned}
& nf(\Rightarrow_{Pre}, g'''(y_{k'''},\varrho_1,\ldots,\varrho_{s'''})) \\
& \quad [y_{k',g'} \leftarrow nf(\Rightarrow_{R_2}, g'(y_{k'},\overline{par}_{f,t}(k',g',1),\ldots,\overline{par}_{f,t}(k',g',s')) \\
& \qquad\qquad\qquad\qquad [y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q]), \\
& \qquad\quad k' \in [q], g' \in G^{(s'+1)}] \\
& \;= \;\text{(by } nf(\Rightarrow_{Pre}, g'''(y_{k'''},\varrho_1,\ldots,\varrho_{s'''})) = y_{k''',g'''} \text{ and substitution)} \\
& nf(\Rightarrow_{R_2}, g'''(y_{k'''},\overline{par}_{f,t}(k''',g''',1),\ldots,\overline{par}_{f,t}(k''',g''',s''')))[y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q]) \\
& \;= \;\text{(by } \varrho_{l'''} = \overline{par}_{f,t}(k''',g''',l''') \text{ for every } l''' \in [s''']) \\
& nf(\Rightarrow_{R_2}, g'''(y_{k'''},\varrho_1,\ldots,\varrho_{s'''})[y_1,\ldots,y_q \leftarrow \theta_1,\ldots,\theta_q]). \quad \square
\end{aligned}$$

For the final proof we need a further auxiliary notion, complementary (in the same sense as the $\overline{par}_{f,t}$- and $\overline{par}_{\phi,(t_1,\ldots,t_p)}$-functions complement each other) to the unary predicate from Definition A.6(1). It indicates, which paths in a right-hand side ϕ of M_1 can be reached by which states of M_2 if the recursion variables in ϕ are instantiated with particular input trees $t_1,\ldots,t_p$.

Definition A.12 ($\vdash_{\phi,(t_1,\ldots,t_p)}$ *as a complementary for* $\vdash_{f,t}$)
For every $r \in \mathbb{N}$, $p \in \mathbb{N}$, $\phi \in RHS(F,\Delta,U_p,Y_r)$, $t_1,\ldots,t_p \in T_\Sigma$, $\pi \in paths(\phi)$ and $g \in G$, we write $\vdash_{\phi,(t_1,\ldots,t_p)} (\pi, g)$ iff there exists a $g' \in G^{(s'+1)}$ such that during a reduction of $g'(mark(\phi)[u_c \leftarrow t_c, c \in [p]], z_1,\ldots,z_{s'})$ with $\Rightarrow_{R_1\cup R_2}$ a call of the form $g(\bullet^\pi,\ldots)$ occurs. $\diamond$

Two "propagation" properties relating the predicates from Definitions A.6(1) and A.12 are proven in the following two lemmata.

Lemma A.13 (*propagation from* $\vdash_{f,\sigma(t_1,\ldots,t_p)}$ *to* $\vdash_{rhs_{f,\sigma},(t_1,\ldots,t_p)}$)
For every $f \in F^{(r+1)}$, $\sigma \in \Sigma^{(p)}$, $t_1,\ldots,t_p \in T_\Sigma$, $k \in [r]$, $g \in G^{(s+1)}$ with $s > 0$, and $\pi \in paths(rhs_{f,\sigma})$: if $\vdash_{f,\sigma(t_1,\ldots,t_p)} (k,g)$ and $lab(rhs_{f,\sigma},\pi) = y_k$, then $\vdash_{rhs_{f,\sigma},(t_1,\ldots,t_p)} (\pi,g)$.

Proof
If $\vdash_{f,\sigma(t_1,\ldots,t_p)} (k,g)$, then there exists $g' \in G^{(s'+1)}$ such that during a reduction of

$$g'(f(\sigma(t_1,\ldots,t_p),y_1,\ldots,y_r),z_1,\ldots,z_{s'})$$

with $\Rightarrow_{R_1\cup R_2}$ a call of the form $g(y_k,\ldots)$ occurs. Clearly, this call must occur during a reduction of $g'(rhs_{f,\sigma}[u_c \leftarrow t_c, c \in [p]], z_1,\ldots,z_{s'})$. Since π is the only path such that $lab(rhs_{f,\sigma},\pi) = y_k$ (because M_1 is non-copying), this corresponds to a call $g(y_k^\pi,\ldots)$ occurring during a reduction of

$$g'(mark(rhs_{f,\sigma})[u_c \leftarrow t_c, c \in [p]], z_1,\ldots,z_{s'})$$

with $\Rightarrow_{R_1\cup R_2}$. This means that $\vdash_{rhs_{f,\sigma},(t_1,\ldots,t_p)} (\pi,g)$. $\square$

Note that in the previous as well as in the following lemma the two preconditions $r > 0$ (implicit from $k \in [r]$ in Lemma A.13) and $s > 0$ cannot be fulfilled in the case that one of the two involved mtts is a tdtt. Hence, the lemmata hold trivially in this case. If those preconditions *are* fulfilled, then neither M_1 nor M_2 is a tdtt, hence we can use the non-copying property of M_1 and the weakly single-use property of M_2.

Lemma A.14 (*propagation from* $\vdash_{\phi,(t_1,\ldots,t_p)}$ *to* $\vdash_{\phi,(t_1,\ldots,t_p)}$ *and* $\vdash_{f,t_i}$)
For every $(r+1) \in rank(F)$ with $r > 0$, $p \in \mathbb{N}$, $\phi \in RHS(F,\Delta,U_p,Y_r)$, $t_1,\ldots,t_p \in T_\Sigma$, $g \in G^{(s+1)}$ with $s > 0$, $j \in \mathbb{N}_+$, $\pi j \in paths(\phi)$ and $lab(\phi,\pi j) \notin U_p$: if $\vdash_{\phi,(t_1,\ldots,t_p)} (\pi j, g)$, then:

1. If $lab(\phi,\pi) = \delta$ for some $\delta \in \Delta^{(q)}$, then for every $g' \in G^{(s'+1)}$: if there is a rule $g'(\delta(v_1,\ldots,v_q),z_1,\ldots,z_{s'}) \to \ldots g(v_j,\ldots)\ldots$ in R_2, then $\vdash_{\phi,(t_1,\ldots,t_p)} (\pi, g')$.
2. If $lab(\phi,\pi) = f$ for some $f \in F^{(q+1)}$, and $lab(\phi,\pi 1) = u_i \in U_p$, then:
   (a) $\vdash_{f,t_i} (j-1, g)$ and
   (b) for every $l \in [s]$, $g' \in G^{(s'+1)}$ and $m \in [s']$: if $z_{g',m} \in Z_G$ occurs in $\overline{par}_{f,t_i}(j-1,g,l)$, then $\vdash_{\phi,(t_1,\ldots,t_p)} (\pi, g')$.

Proof
If $\vdash_{\phi,(t_1,\ldots,t_p)} (\pi j, g)$, then there exists $g'' \in G^{(s''+1)}$ such that during a reduction of $g''(mark(\phi)[u_c \leftarrow t_c, c \in [p]], z_1,\ldots,z_{s''})$ with $\Rightarrow_{R_1 \cup R_2}$ a call of the form $g(\bullet^{\pi j},\ldots)$ occurs. By Lemma A.3 there exists $g''' \in G^{(s'''+1)}$ such that

$$\vdash_{\phi,(t_1,\ldots,t_p)} (\pi, g''')$$

and reduction of a call of the form $g'''(sub(mark(\phi),\pi)[u_c \leftarrow t_c, c \in [p]],\ldots)$ leads to the mentioned $g(\bullet^{\pi j},\ldots)$-call.

1. If $lab(\phi,\pi) = \delta$, then we know that a reduction of $g'''(\delta^\pi(\ldots),\ldots)$ leads to a $g(\bullet^{\pi j},\ldots)$-call. Hence, there must be a call of g on v_j in a rule of g''' at δ. But by the weakly single-use property of M_2 and the rule assumed for g' at δ, this implies $g''' = g'$, and hence $\vdash_{\phi,(t_1,\ldots,t_p)} (\pi, g')$.
2. If $lab(\phi,\pi) = f$ and $lab(\phi,\pi 1) = u_i$, then we know that a reduction of $g'''(f^\pi(t_i,\ldots),\ldots)$ leads to a $g(\bullet^{\pi j},\ldots)$-call. This means that a call of g reaches the $(j-1)$st context parameter position of the call of f^π on t_i, hence $\vdash_{f,t_i} (j-1,g)$. By Theorem A.4 we have for every $l \in [s]$:

   $$\overline{par}_{f,t_i}(j-1,g,l) \in RHS(G,\Omega,Y_q,\{z_{g''',1},\ldots,z_{g''',s'''}\}).$$

   Hence, if a $z_{g',m}$ occurs in $\overline{par}_{f,t_i}(j-1,g,l)$, then $g''' = g'$, which implies $\vdash_{\phi,(t_1,\ldots,t_p)} (\pi, g')$. □

Now, we are going to present and prove the main lemma of this appendix. It consists of six statements that will be proven by nested simultaneous induction. In order to illustrate the proof structure, the statements are arranged in a nested fashion. We have a simultaneous induction of statements **I** and **II** for trees over the ranked alphabet Σ. Statements **I**a and **I**b specify exactly the semantics of the (f,g)- and (k_f,l_g)-states from Construction 5.1. Statement **II**b makes precise how the par_ϕ-functions (computing in ignorance of concrete input trees for the recursion variables in M_1's right-hand side ϕ) are related to the $\overline{par}_{\phi,(t_1,\ldots,t_p)}$-functions from Definition A.1. Nested inside the statement **II**a, we have statements **II**(a)**i** and **II**(a)**ii** that will also be proven by simultaneous induction, but for all possible right-hand sides built from states and output symbols of M_1 instead of all input trees. Since these technical statements are only subsidiary (apart from the fact that indeed statement **II**(a)**ii**(A) will be used in the main Theorem A.16), we will not

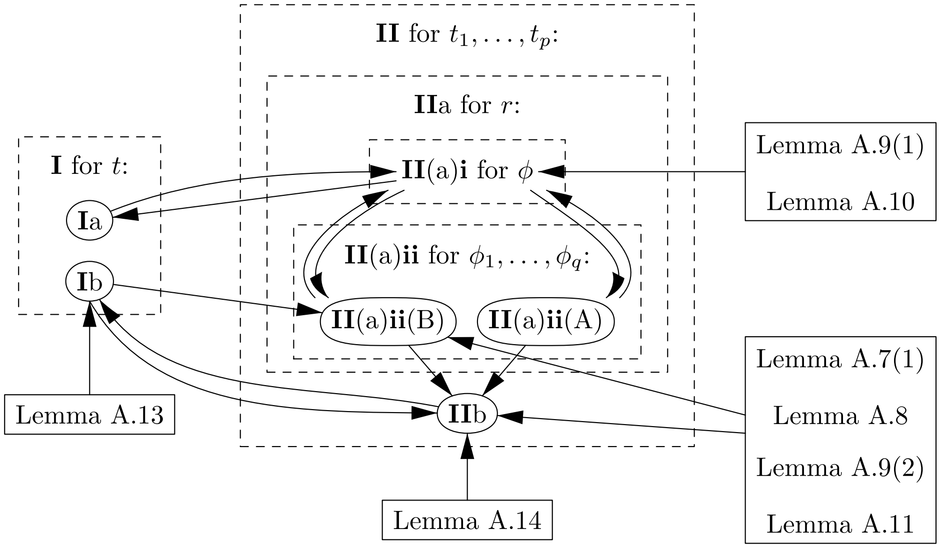

Fig. A 2. Proof structure for Lemma A.15.

try to provide an intuition for them. The proof structure and dependencies used in the inductive argument are shown in Fig. A 2.

Lemma A.15 (*relating ingredients of* $M_{1;2}$ *with* $\overline{par}_{f,t}$*- and* $\overline{par}_{\phi,(t_1,\dots,t_p)}$*-functions*)

I. For every $t \in T_\Sigma$:

(a) for every $f \in F^{(r+1)}$ and $g \in G^{(s+1)}$:
$nf(\Rightarrow_{R_{1;2}}, (f,g)(t, y_{1,g_1}, \dots, y_{r,g_\mu}, z_1, \dots, z_s))$
$= nf(\Rightarrow_{Pre}, nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(t, y_1, \dots, y_r)), z_1, \dots, z_s)))$

(b) for every $f \in F^{(r+1)}$, $g \in G^{(s+1)}$, $k \in [r]$ and $l \in [s]$, if $\vdash_{f,t} (k,g)$ then:
$nf(\Rightarrow_{R_{1;2}}, (k_f, l_g)(t, y_{1,g_1}, \dots, y_{r,g_\mu}, z_{g_1,1}, \dots, z_{g_\mu,s_\mu}))$
$= nf(\Rightarrow_{Pre}, \overline{par}_{f,t}(k,g,l))$

II. For every $p \in \mathbb{N}, t_1, \dots, t_p \in T_\Sigma$:

(a) for every $r \in \mathbb{N}$:

i For every $\phi \in RHS(F, \Delta, U_p, Y_r)$:
for every $g \in G^{(s+1)}$:
$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, g(\phi, z_1, \dots, z_s))[u_1, \dots, u_p \leftarrow t_1, \dots, t_p])$
$= nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, \phi[u_1, \dots, u_p \leftarrow t_1, \dots, t_p]), z_1, \dots, z_s))$

ii For every $q \in \mathbb{N}, \phi_1, \dots, \phi_q \in RHS(F, \Delta, U_p, Y_r)$:

(A) for every $s \in \mathbb{N}$ and $\psi \in RHS(G, \Omega, V_q, Z_s)$:
$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, \psi[v_d \leftarrow \phi_d, d \in [q]])[u_c \leftarrow t_c, c \in [p]])$
$= nf(\Rightarrow_{R_2}, \psi[v_d \leftarrow nf(\Rightarrow_{R_1}, \phi_d[u_1, \dots, u_p \leftarrow t_1, \dots, t_p]), d \in [q]])$

(B) for every $i \in [p]$, $f \in F^{(q+1)}$, $\mathcal{C} \subseteq [q] \times G$, $k \in [q]$ and $g \in G^{(s+1)}$:

if $\vdash_{f,t_i} (k, g)$ and not $(k, g) \prec^*_{f,t_i} \mathcal{C}$, then:
$$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, nest_f(k, g, \mathcal{C})[y'_1, \ldots, y'_q \leftarrow \phi_1, \ldots, \phi_q]) [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p])$$
$$= nf(\Rightarrow_{R_2}, g(y_k, \overline{par}_{f,t_i}(k, g, 1), \ldots, \overline{par}_{f,t_i}(k, g, s)) [y_b \leftarrow nf(\Rightarrow_{R_1}, \phi_b[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]), b \in [q]])$$

(b) for every $(r + 1) \in rank(F)$ with $r > 0$, $\phi \in RHS(F, \Delta, U_p, Y_r)$, $\pi \in paths(\phi)$ such that $lab(\phi, \pi) \notin U_p$, for every $g \in G^{(s+1)}$ and $l \in [s]$:
if $\vdash_{\phi,(t_1,\ldots,t_p)} (\pi, g)$, then:
$$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, par_\phi(\pi, g, l))[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p])$$
$$= \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g, l)$$

Proof
We prove **I** and **II** by simultaneous induction:

I $\Leftarrow$ II : Let $t = \sigma(t_1, \ldots, t_p)$ for some $\sigma \in \Sigma^{(p)}$ and $t_1, \ldots, t_p \in T_\Sigma$.

Ia :
$$nf(\Rightarrow_{R_{1;2}}, (f, g)(\sigma(t_1, \ldots, t_p), y_{1,g_1}, \ldots, y_{r,g_\mu}, z_1, \ldots, z_s))$$
$= \;$ (by $\Rightarrow_{R_{1;2}}$)
$$nf(\Rightarrow_{R_{1;2}}, rhs_{(f,g),\sigma}[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p])$$
$= \;$ (by definition of $rhs_{(f,g),\sigma}$ in Construction 5.1)
$$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pre \cup Pair}, g(rhs_{f,\sigma}, z_1, \ldots, z_s))[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p])$$
$= \;$ (by reordering of normal form computations and substitution)
$$nf(\Rightarrow_{Pre}, nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, g(rhs_{f,\sigma}, z_1, \ldots, z_s)) [u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]))$$
$= \;$ (by induction hypothesis **II**(a)**i** for $\phi = rhs_{f,\sigma} \in RHS(F, \Delta, U_p, Y_r)$)
$$nf(\Rightarrow_{Pre}, nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, rhs_{f,\sigma}[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]), z_1, \ldots, z_s)))$$
$= \;$ (by $\Rightarrow_{R_1}$)
$$nf(\Rightarrow_{Pre}, nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(\sigma(t_1, \ldots, t_p), y_1, \ldots, y_r)), z_1, \ldots, z_s)))$$

Ib : Assume $\vdash_{f,\sigma(t_1,\ldots,t_p)} (k, g)$. We have by $\Rightarrow_{R_{1;2}}$:
$$nf(\Rightarrow_{R_{1;2}}, (k_f, l_g)(\sigma(t_1, \ldots, t_p), y_{1,g_1}, \ldots, y_{r,g_\mu}, z_{g_1,1}, \ldots, z_{g_\mu,s_\mu}))$$
$$= nf(\Rightarrow_{R_{1;2}}, rhs_{(k_f,l_g),\sigma}[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]).$$

Now, there are two possible cases:

1. If $rhs_{f,\sigma}$ does *not* contain the context variable y_k, then by Construction 5.1 we have $rhs_{(k_f,l_g),\sigma} = nf(\Rightarrow_{R_2 \cup Pre \cup Pair}, nil) = nil$ and thus:
$$nf(\Rightarrow_{R_{1;2}}, rhs_{(k_f,l_g),\sigma}[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p])$$
$= \;$ (by $rhs_{(k_f,l_g),\sigma} = nil = nf(\Rightarrow_{Pre}, nil)$)
$$nf(\Rightarrow_{Pre}, nil)$$
$= \;$ (by definition of $\overline{par}_{f,\sigma(t_1,\ldots,t_p)}$)
$$nf(\Rightarrow_{Pre}, \overline{par}_{f,\sigma(t_1,\ldots,t_p)}(k, g, l)).$$
2. If $rhs_{f,\sigma} \in RHS(F, \Delta, U_p, Y_r)$ *does* contain the context variable y_k, and the path of the unique such occurrence (notice that M_1 is non-copying) is $\pi_{y_k} \in paths(rhs_{f,\sigma})$, then we have:

$$nf(\Rightarrow_{R_{1;2}}, rhs_{(k_f,l_g),\sigma}[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p])$$
$$= \text{(by definition of } rhs_{(k_f,l_g),\sigma} \text{ in Construction 5.1)}$$
$$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2\cup Pre\cup Pair}, par_{rhs_{f,\sigma}}(\pi_{y_k}, g, l))[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p])$$
$$= \text{(by reordering of normal form computations and substitution)}$$
$$nf(\Rightarrow_{Pre}, nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2\cup Pair}, par_{rhs_{f,\sigma}}(\pi_{y_k}, g, l))[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p])).$$

By $\vdash_{f,\sigma(t_1,\ldots,t_p)} (k,g)$ and Lemma A.13[2], we know $\vdash_{rhs_{f,\sigma},(t_1,\ldots,t_p)} (\pi_{y_k}, g)$, and hence by induction hypothesis **II**b[3] for $\phi = rhs_{f,\sigma}$, the previous expression is equal to:

$$nf(\Rightarrow_{Pre}, \overline{par}_{rhs_{f,\sigma},(t_1,\ldots,t_p)}(\pi_{y_k}, g, l))$$
$$= \text{(by definition of } \overline{par}_{f,\sigma(t_1,\ldots,t_p)})$$
$$nf(\Rightarrow_{Pre}, \overline{par}_{f,\sigma(t_1,\ldots,t_p)}(k, g, l)).$$

II $\Leftarrow$ **I :** We first prove statement **II**a and then use it for proving **II**b. In doing so, we can assume statement **I** for the fixed $t_1,\ldots,t_p$.

IIa : by simultaneous induction of **II**(a)**i** and **II**(a)**ii**:

II(a)i $\Leftarrow$ **II(a)ii :** by case analysis on $\phi \in RHS(F,\Delta,U_p,Y_r)$:

$\underline{\phi = y_k \in Y_r}$:

Clearly both sides of the equation are equal to $g(y_k, z_1,\ldots,z_s)$.

$\underline{\phi = \delta(\phi_1,\ldots,\phi_q)}$ for some $\delta \in \Delta^{(q)}$ and $\phi_1,\ldots,\phi_q \in RHS(F,\Delta,U_p,Y_r)$:

$$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2\cup Pair}, g(\delta(\phi_1,\ldots,\phi_q), z_1,\ldots,z_s))[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p])$$
$$= \text{(with rule } g(\delta(v_1,\ldots,v_q), z_1,\ldots,z_s) \to rhs_{g,\delta} \text{ in } R_2)$$
$$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2\cup Pair}, rhs_{g,\delta}[v_1,\ldots,v_q \leftarrow \phi_1,\ldots,\phi_q])[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p])$$
$$= \text{(by induction hypothesis } \mathbf{II}\text{(a)}\mathbf{ii}\text{(A) for } \psi = rhs_{g,\delta})$$
$$nf(\Rightarrow_{R_2}, rhs_{g,\delta}[v_d \leftarrow nf(\Rightarrow_{R_1}, \phi_d[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p]), d \in [q]])$$
$$= \text{(by the above rule in } R_2)$$
$$nf(\Rightarrow_{R_2}, g(\delta(nf(\Rightarrow_{R_1}, \phi_1[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p]), \ldots, nf(\Rightarrow_{R_1}, \phi_q[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p])), z_1,\ldots,z_s))$$
$$= \text{(by substitution and normal form)}$$
$$nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, \delta(\phi_1,\ldots,\phi_q)[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p]), z_1,\ldots,z_s))$$

$\underline{\phi = f(u_i,\phi_1,\ldots,\phi_q)}$ for $f \in F^{(q+1)}, u_i \in U_p, \phi_1,\ldots,\phi_q \in RHS(F,\Delta,U_p,Y_r)$:

[2] Note that the precondition $s > 0$ is fulfilled, since $l \in [s]$.
[3] Note that the precondition $r > 0$ is fulfilled, since $k \in [r]$.

By definition of *Pair* we have:

$$
\begin{aligned}
& nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, g(f(u_i, \phi_1, \ldots, \phi_q), z_1, \ldots, z_s)) \\
& \qquad [u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]) \\
& = nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, (f,g)(u, nest_f(1, g_1, \emptyset), \\
& \qquad \ldots, \\
& \qquad nest_f(q, g_\mu, \emptyset), z_1, \ldots, z_s) \\
& \qquad [z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s] \\
& \qquad [z_{g_1,1}, \ldots, z_{g_\mu,s_\mu} \leftarrow nil, \ldots, nil] \\
& \qquad [u, y'_1, \ldots, y'_q \leftarrow u_i, \phi_1, \ldots, \phi_q]) \\
& \qquad [u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]).
\end{aligned}
$$

By composition of substitutions, this is equivalent to:

$$
\begin{aligned}
& nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, (f,g)(u_i, nest_f(1, g_1, \emptyset)[y'_b \leftarrow \phi_b, b \in [q]] \\
& \qquad [u \leftarrow u_i][z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s] \\
& \qquad [z_{g_1,1}, \ldots, z_{g_\mu,s_\mu} \leftarrow nil, \ldots, nil], \\
& \qquad \ldots, \\
& \qquad nest_f(q, g_\mu, \emptyset)[y'_b \leftarrow \phi_b, b \in [q]] \\
& \qquad [u \leftarrow u_i][z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s] \\
& \qquad [z_{g_1,1}, \ldots, z_{g_\mu,s_\mu} \leftarrow nil, \ldots, nil], \\
& \qquad z_1, \ldots, z_s)) \\
& \qquad [u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]) \\
& = \text{(by reordering of substitutions and normal form computation)} \\
& nf(\Rightarrow_{R_{1;2}}, (f,g)(t_i, nf(\Rightarrow_{R_2 \cup Pair}, nest_f(1, g_1, \emptyset)[y'_b \leftarrow \phi_b, b \in [q]]) \\
& \qquad [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p] \\
& \qquad [z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s] \\
& \qquad [z_{g_1,1}, \ldots, z_{g_\mu,s_\mu} \leftarrow nil, \ldots, nil], \\
& \qquad \ldots, \\
& \qquad nf(\Rightarrow_{R_2 \cup Pair}, nest_f(q, g_\mu, \emptyset)[y'_b \leftarrow \phi_b, b \in [q]]) \\
& \qquad [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p] \\
& \qquad [z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s] \\
& \qquad [z_{g_1,1}, \ldots, z_{g_\mu,s_\mu} \leftarrow nil, \ldots, nil], \\
& \qquad z_1, \ldots, z_s)) \\
& = \text{(by reordering of normal form computation and substitution)} \\
& nf(\Rightarrow_{R_{1;2}}, (f,g)(t_i, y_{1,g_1}, \ldots, y_{q,g_\mu}, z_1, \ldots, z_s)) \\
& \quad [y_{k,g'} \leftarrow nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, nest_f(k, g', \emptyset)[y'_b \leftarrow \phi_b, b \in [q]]) \\
& \qquad [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]) \\
& \qquad [z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s][z_{g_1,1}, \ldots, z_{g_\mu,s_\mu} \leftarrow nil, \ldots, nil], \\
& \qquad k \in [q], g' \in G^{(s'+1)}] \\
& = \text{(by induction hypothesis } \mathbf{I}\text{a for } t_i) \\
& nf(\Rightarrow_{Pre}, nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(t_i, y_1, \ldots, y_q)), z_1, \ldots, z_s))) \\
& \quad [y_{k,g'} \leftarrow nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, nest_f(k, g', \emptyset)[y'_b \leftarrow \phi_b, b \in [q]]) \\
& \qquad [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]) \\
& \qquad [z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s][z_{g_1,1}, \ldots, z_{g_\mu,s_\mu} \leftarrow nil, \ldots, nil], \\
& \qquad k \in [q], g' \in G^{(s'+1)}].
\end{aligned}
$$

For every $y_{k,g'}$ occurring in

$$nf(\Rightarrow_{Pre}, nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(t_i, y_1, \ldots, y_q)), z_1, \ldots, z_s))) ,$$

Lemma A.9(1) implies that $\vdash_{f,t_i} (k, g')$. Further, it holds trivially that not $(k, g') \prec^*_{f,t_i} \emptyset$. Hence, induction hypothesis **II**(a)**ii**(B) can be applied to the normal forms replaced for the $y_{k,g'}$ in the previous expression, giving the equivalent:

$$\begin{aligned}
& nf(\Rightarrow_{Pre}, nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(t_i, y_1, \ldots, y_q)), z_1, \ldots, z_s))) \\
& \quad [y_{k,g'} \leftarrow nf(\Rightarrow_{R_2}, g'(y_k, \overline{par}_{f,t_i}(k, g', 1), \ldots, \overline{par}_{f,t_i}(k, g', s')) \\
& \qquad\qquad\qquad [y_b \leftarrow nf(\Rightarrow_{R_1}, \phi_b[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]), \\
& \qquad\qquad\qquad\quad b \in [q]]) \\
& \qquad\qquad [z_{g,1}, \ldots, z_{g,s} \leftarrow z_1, \ldots, z_s][z_{g_1,1}, \ldots, z_{g_\mu,s_\mu} \leftarrow nil, \ldots, nil], \\
& \qquad k \in [q], g' \in G^{(s'+1)}] \\
& = \text{(by Lemma A.10, where for every } b \in [q]\text{:} \\
& \qquad \theta_b = nf(\Rightarrow_{R_1}, \phi_b[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]) \in T_\Delta(Y_r)) \\
& nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(t_i, nf(\Rightarrow_{R_1}, \phi_1[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]), \\
& \qquad\qquad\qquad\qquad \ldots, \\
& \qquad\qquad\qquad nf(\Rightarrow_{R_1}, \phi_q[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]))), \\
& \qquad z_1, \ldots, z_s)) \\
& = \text{(by substitution and by confluence of } \Rightarrow_{R_1}) \\
& nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, f(u_i, \phi_1, \ldots, \phi_q)[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]), \\
& \qquad z_1, \ldots, z_s)).
\end{aligned}$$

II(a)ii $\Leftarrow$ II(a)i :

II(a)ii(A) : straightforward structural induction on $\psi \in RHS(G, \Omega, V_q, Z_s)$, using induction hypothesis **II**(a)**i** on ϕ_j in the case $\psi = g(v_j, \ldots)$ for $j \in [q]$.

II(a)ii(B) : For fixed $i \in [p]$ and $f \in F^{(q+1)}$, we prove the statement for every $\mathcal{C} \in \mathcal{P}([q] \times G)$ by induction on the reversed subset-order:

- $\mathcal{C} = [q] \times G$:
  In this base case we have $(k, g) \prec^*_{f,t_i} \mathcal{C}$ for every $k \in [q]$ and $g \in G$, thus nothing remains to prove.

- $\mathcal{C} \subset [q] \times G$:
  Assume $\vdash_{f,t_i} (k, g)$. If not $(k, g) \prec^*_{f,t_i} \mathcal{C}$, then $(k, g) \notin \mathcal{C}$ and thus:

$$\begin{aligned}
& nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, nest_f(k, g, \mathcal{C})[y'_1, \ldots, y'_q \leftarrow \phi_1, \ldots, \phi_q]) \\
& \qquad\qquad [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]) \\
& \quad = \text{(by definition of } nest_f)
\end{aligned}$$

$$
\begin{aligned}
& nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, g(y'_k, (k_f, 1_g)(u, nest_f(1, g_1, \mathcal{C} \cup \{(k,g)\}), \\
& \qquad \ldots, \\
& \qquad nest_f(q, g_\mu, \mathcal{C} \cup \{(k,g)\}), \\
& \qquad z_{g_1,1}, \ldots, z_{g_\mu,s_\mu}), \\
& \qquad \ldots, \\
& \qquad (k_f, s_g)(u, nest_f(1, g_1, \mathcal{C} \cup \{(k,g)\}), \\
& \qquad \ldots, \\
& \qquad nest_f(q, g_\mu, \mathcal{C} \cup \{(k,g)\}), \\
& \qquad z_{g_1,1}, \ldots, z_{g_\mu,s_\mu})) \\
& \qquad [y'_1, \ldots, y'_q \leftarrow \phi_1, \ldots, \phi_q]) \\
& \qquad [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p])
\end{aligned}
$$

= (by substitution and reordering of normal form computation)

$$
\begin{aligned}
& nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, g(\phi_k, z_1, \ldots, z_s)) \\
& \quad [z_l \leftarrow (k_f, l_g)(u, nf(\Rightarrow_{R_2 \cup Pair}, nest_f(1, g_1, \mathcal{C} \cup \{(k,g)\}) \\
& \qquad [y'_b \leftarrow \phi_b, b \in [q]]), \\
& \qquad \ldots, \\
& \qquad nf(\Rightarrow_{R_2 \cup Pair}, nest_f(q, g_\mu, \mathcal{C} \cup \{(k,g)\}) \\
& \qquad [y'_b \leftarrow \phi_b, b \in [q]]), \\
& \qquad z_{g_1,1}, \ldots, z_{g_\mu,s_\mu}), \\
& \qquad l \in [s]] \\
& \quad [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p])
\end{aligned}
$$

= (by reordering of substitutions and normal form computation)

$$
\begin{aligned}
& nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, g(\phi_k, z_1, \ldots, z_s))[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]) \\
& \quad [z_l \leftarrow nf(\Rightarrow_{R_{1;2}}, (k_f, l_g)(u_i, \\
& \qquad nf(\Rightarrow_{R_2 \cup Pair}, nest_f(1, g_1, \mathcal{C} \cup \{(k,g)\})[y'_b \leftarrow \phi_b, b \in [q]]) \\
& \qquad [u \leftarrow u_i], \\
& \qquad \ldots, \\
& \qquad nf(\Rightarrow_{R_2 \cup Pair}, nest_f(q, g_\mu, \mathcal{C} \cup \{(k,g)\})[y'_b \leftarrow \phi_b, b \in [q]]) \\
& \qquad [u \leftarrow u_i], \\
& \qquad z_{g_1,1}, \ldots, z_{g_\mu,s_\mu})[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]), \\
& \qquad l \in [s]]
\end{aligned}
$$

= (by induction hypothesis **II**(a)**i** for ϕ_k)

$$
\begin{aligned}
& nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, \phi_k[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]), z_1, \ldots, z_s)) \\
& \quad [z_l \leftarrow nf(\Rightarrow_{R_{1;2}}, (k_f, l_g)(u_i, \\
& \qquad nf(\Rightarrow_{R_2 \cup Pair}, nest_f(1, g_1, \mathcal{C} \cup \{(k,g)\})[y'_b \leftarrow \phi_b, b \in [q]]) \\
& \qquad [u \leftarrow u_i], \\
& \qquad \ldots, \\
& \qquad nf(\Rightarrow_{R_2 \cup Pair}, nest_f(q, g_\mu, \mathcal{C} \cup \{(k,g)\})[y'_b \leftarrow \phi_b, b \in [q]]) \\
& \qquad [u \leftarrow u_i], \\
& \qquad z_{g_1,1}, \ldots, z_{g_\mu,s_\mu})[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]), \\
& \qquad l \in [s]].
\end{aligned}
$$

We name this expression **(+)** and continue to calculate on the expressions substituted for the z_l with $l \in [s]$:

$$nf(\Rightarrow_{R_{1;2}}, (k_f, l_g)(u_i, \\ nf(\Rightarrow_{R_2 \cup Pair}, nest_f(1, g_1, \mathcal{C} \cup \{(k,g)\})[y'_b \leftarrow \phi_b, b \in [q]])[u \leftarrow u_i], \\ \ldots, \\ nf(\Rightarrow_{R_2 \cup Pair}, nest_f(q, g_\mu, \mathcal{C} \cup \{(k,g)\})[y'_b \leftarrow \phi_b, b \in [q]])[u \leftarrow u_i], \\ z_{g_1,1}, \ldots, z_{g_\mu,s_\mu})[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p])$$

$=$ (by substitution and reordering of normal form computation)

$$nf(\Rightarrow_{R_{1;2}}, (k_f, l_g)(t_i, y_{1,g_1}, \ldots, y_{q,g_\mu}, z_{g_1,1}, \ldots, z_{g_\mu,s_\mu})) \\ [y_{k',g'} \leftarrow nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, nest_f(k', g', \mathcal{C} \cup \{(k,g)\}) \\ [y'_b \leftarrow \phi_b, b \in [q]]) \\ [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]), \\ k' \in [q], g' \in G^{(s'+1)}]$$

$=$ (by induction hypothesis **I**b for t_i, using the prerequisite $\vdash_{f,t_i} (k, g)$)

$$nf(\Rightarrow_{Pre}, \overline{par}_{f,t_i}(k, g, l)) \\ [y_{k',g'} \leftarrow nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, nest_f(k', g', \mathcal{C} \cup \{(k,g)\}) \\ [y'_b \leftarrow \phi_b, b \in [q]]) \\ [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]), \\ k' \in [q], g' \in G^{(s'+1)}].$$

Now, we want to use the induction hypothesis on $\mathcal{C} \cup \{(k,g)\}$, as expressed below[4], in order to show that for every $l \in [s]$ this is equivalent to the following expression **(++)**:

$$nf(\Rightarrow_{Pre}, \overline{par}_{f,t_i}(k, g, l)) \\ [y_{k',g'} \leftarrow nf(\Rightarrow_{R_2}, g'(y_{k'}, \overline{par}_{f,t_i}(k', g', 1), \ldots, \overline{par}_{f,t_i}(k', g', s')) \\ [y_b \leftarrow nf(\Rightarrow_{R_1}, \phi_b[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]), b \in [q]]), \\ k' \in [q], g' \in G^{(s'+1)}].$$

To this aim, we need to establish that for every $y_{k',g'}$ occurring in $nf(\Rightarrow_{Pre}, \overline{par}_{f,t_i}(k, g, l))$ we have:

$$\vdash_{f,t_i} (k', g') \text{ and not } (k', g') \prec^*_{f,t_i} \mathcal{C} \cup \{(k,g)\}.$$

For every such a $y_{k',g'}$ we have $(k, g) \prec_{f,t_i} (k', g')$ by Lemma A.9(2). Hence, the required $\vdash_{f,t_i} (k', g')$ follows from the prerequisite $\vdash_{f,t_i} (k, g)$ by Lemma A.7(1). Furthermore, if $(k', g') \prec^*_{f,t_i} \mathcal{C} \cup \{(k,g)\}$ would hold, then we would also have $(k, g) \prec^+_{f,t_i} \mathcal{C} \cup \{(k,g)\}$, which cannot be true, because by the assumption at the very beginning of the present case we know that not $(k, g) \prec^*_{f,t_i} \mathcal{C}$ and by Lemma A.8 and $\vdash_{f,t_i} (k, g)$ we know that not $(k, g) \prec^+_{f,t_i} (k, g)$.

Let us now return to expression **(+)**. Replacing the expression substi-

[4] For every $k' \in [q]$ and $g' \in G^{(s'+1)}$, if $\vdash_{f,t_i} (k', g')$ and not $(k', g') \prec^*_{f,t_i} \mathcal{C} \cup \{(k,g)\}$, then:

$$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, nest_f(k', g', \mathcal{C} \cup \{(k,g)\})[y'_b \leftarrow \phi_b, b \in [q]]) \\ [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]) \\ = nf(\Rightarrow_{R_2}, g'(y_{k'}, \overline{par}_{f,t_i}(k', g', 1), \ldots, \overline{par}_{f,t_i}(k', g', s')) \\ [y_b \leftarrow nf(\Rightarrow_{R_1}, \phi_b[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]), b \in [q]])$$

tuted there for z_l (with $l \in [s]$) by the expression **(++)**—shown above to be equivalent—gives us the following equivalent of **(+)**:

$$
\begin{aligned}
& nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, \phi_k[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p]), z_1,\ldots,z_s)) \\
& \quad [z_l \leftarrow nf(\Rightarrow_{Pre}, \overline{par}_{f,t_i}(k,g,l)) \\
& \qquad\qquad [y_{k',g'} \leftarrow nf(\Rightarrow_{R_2}, g'(y_{k'}, \overline{par}_{f,t_i}(k',g',1),\ldots,\overline{par}_{f,t_i}(k',g',s')) \\
& \qquad\qquad\qquad\qquad [y_b \leftarrow nf(\Rightarrow_{R_1}, \phi_b[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p]), \\
& \qquad\qquad\qquad\qquad\quad b \in [q]]), \\
& \qquad\qquad\quad k' \in [q], g' \in G^{(s'+1)}], \\
& \qquad l \in [s]]
\end{aligned}
$$

$=$ (by Lemma A.11 on the normal forms replaced for $z_1,\ldots,z_s$, using that $\vdash_{f,t_i} (k,g)$, where for every $b \in [q]$: $\theta_b = nf(\Rightarrow_{R_1}, \phi_b[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p]) \in T_\Delta(Y_r)$)

$$
\begin{aligned}
& nf(\Rightarrow_{R_2}, g(nf(\Rightarrow_{R_1}, \phi_k[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p]), z_1,\ldots,z_s)) \\
& \quad [z_l \leftarrow nf(\Rightarrow_{R_2}, \overline{par}_{f,t_i}(k,g,l) \\
& \qquad\qquad\qquad [y_b \leftarrow nf(\Rightarrow_{R_1}, \phi_b[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p]), b \in [q]]), \\
& \qquad l \in [s]]
\end{aligned}
$$

$=$ (by reordering of normal form computation and by substitution)

$$
\begin{aligned}
& nf(\Rightarrow_{R_2}, g(y_k, \overline{par}_{f,t_i}(k,g,1),\ldots,\overline{par}_{f,t_i}(k,g,s)) \\
& \qquad [y_b \leftarrow nf(\Rightarrow_{R_1}, \phi_b[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p]), b \in [q]]).
\end{aligned}
$$

IIb : In proving **II**b we may use the statements from **II**(a)**ii**, which have previously been proven for every $q \in \mathbb{N}$, $\phi_1,\ldots,\phi_q \in RHS(F,\Delta,U_p,Y_r)$, but for the fixed $t_1,\ldots,t_p$ under consideration in **II**b. For fixed $(r+1) \in rank(F)$ with $r > 0$, and $\phi \in RHS(F,\Delta,U_p,Y_r)$, we proceed by induction on the prefix-order of paths in ϕ:

- In the base case ε both sides of the equation are equal to $z_{g,l}$ by the definitions of par_ϕ and $\overline{par}_{\phi,(t_1,\ldots,t_p)}$.
- The inductive case $\pi j \in paths(\phi)$ with $j \in \mathbb{N}_+$ and $lab(\phi,\pi j) \notin U_p$ is shown by case distinction on $lab(\phi,\pi)$, assuming $\vdash_{\phi,(t_1,\ldots,t_p)} (\pi j, g)$:

  $\underline{lab(\phi,\pi) = \delta}$ for some $\delta \in \Delta^{(q)}$ and $j \in [q]$:

  — If there is no $g(v_j,\ldots)$-call in the δ-rules of M_2, then both sides of the equation are equal to nil by the definitions of par_ϕ and $\overline{par}_{\phi,(t_1,\ldots,t_p)}$.

  — Otherwise, if the unique such call (notice that M_2 is weakly single-use) looks, with $g' \in G^{(s'+1)}$ and $\psi_1,\ldots,\psi_s \in RHS(G,\Omega,V_q,Z_{s'})$, as follows:

  $$g'(\delta(v_1,\ldots,v_q), z_1,\ldots,z_{s'}) \to \ldots g(v_j,\psi_1,\ldots,\psi_s)\ldots\ ,$$

  then we can calculate:

  $$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2\cup Pair}, par_\phi(\pi j,g,l))[u_1,\ldots,u_p \leftarrow t_1,\ldots,t_p])$$

  $=$ (by definition of par_ϕ)

  $$
  \begin{aligned}
  & nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2\cup Pair}, \\
  & \qquad\qquad \psi_l[v_1,\ldots,v_q \leftarrow sub(\phi,\pi 1),\ldots,sub(\phi,\pi q), \\
  & \qquad\qquad\quad z_1,\ldots,z_{s'} \leftarrow par_\phi(\pi,g',1),\ldots,par_\phi(\pi,g',s')])
  \end{aligned}
  $$

$$[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p])$$
$$= \text{(by reordering of normal form computation and substitutions)}$$
$$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, \psi_l[v_1, \ldots, v_q \leftarrow sub(\phi, \pi 1), \ldots, sub(\phi, \pi q)] [u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p] [z_m \leftarrow par_\phi(\pi, g', m)[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p], m \in [s']]))$$
$$= \text{(by reordering of substitution and normal form computations)}$$
$$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, \psi_l[v_1, \ldots, v_q \leftarrow sub(\phi, \pi 1), \ldots, sub(\phi, \pi q)]) [u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]) [z_m \leftarrow nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, par_\phi(\pi, g', m)) [u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]), m \in [s']].$$

From $\vdash_{\phi,(t_1,\ldots,t_p)} (\pi j, g)$ and the rule of g' at δ given above follows by Lemma A.14(1) that $\vdash_{\phi,(t_1,\ldots,t_p)} (\pi, g')$. Hence, we can use the induction hypothesis for π to replace the expressions substituted for the z_m, yielding the following equivalent:

$$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, \psi_l[v_1, \ldots, v_q \leftarrow sub(\phi, \pi 1), \ldots, sub(\phi, \pi q)]) [u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]) [z_m \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g', m), m \in [s']]$$
$$= \text{(by the previously proven statement } \mathbf{II}\text{(a)}\mathbf{ii}\text{(A) for } sub(\phi, \pi 1), \ldots, sub(\phi, \pi q) \in RHS(F, \Delta, U_p, Y_r), s' + 1 \in rank(G) \text{ and } \psi_l \in RHS(G, \Omega, V_q, Z_{s'}))$$
$$nf(\Rightarrow_{R_2}, \psi_l[v_d \leftarrow nf(\Rightarrow_{R_1}, sub(\phi, \pi d)[u_c \leftarrow t_c, c \in [p]]), d \in [q]]) [z_m \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g', m), m \in [s']]$$
$$= \text{(since the } \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g', m) \text{ for } m \in [s'] \text{ are } \Rightarrow_{R_2}\text{-normal forms, by the remark below Definition A.1)}$$
$$nf(\Rightarrow_{R_2}, \psi_l[v_d \leftarrow nf(\Rightarrow_{R_1}, sub(\phi, \pi d)[u_c \leftarrow t_c, c \in [p]]), d \in [q]] [z_m \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g', m), m \in [s']])$$
$$= \text{(by composing substitutions, using that for every } m \in [s'] \text{ and } d \in [q] \text{, no } z_m \text{ occurs in } nf(\Rightarrow_{R_1}, sub(\phi, \pi d)[u_c \leftarrow t_c, c \in [p]]))$$
$$nf(\Rightarrow_{R_2}, \psi_l[v_d \leftarrow nf(\Rightarrow_{R_1}, sub(\phi, \pi d)[u_c \leftarrow t_c, c \in [p]]), d \in [q], z_m \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g', m), m \in [s']])$$
$$= \text{(by definition of } \overline{par}_{\phi,(t_1,\ldots,t_p)} \text{, taking into account the unique } g(v_j, \ldots)\text{-call in the } \delta\text{-rules of } M_2 \text{ as given above)}$$
$$\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi j, g, l).$$

<u>$lab(\phi, \pi) = f$</u> for some $f \in F^{(q+1)}$, $1 \leq j - 1 \leq q$ and $lab(\phi, \pi 1) = u_i \in U_p$:

$$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, par_\phi(\pi j, g, l))[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p])$$
$$= \text{(by definition of } par_\phi)$$
$$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair},$$

$$
\begin{aligned}
&((j-1)_f, l_g)(u, nest_f(1, g_1, \{(j-1,g)\}),\\
&\qquad \ldots,\\
&\qquad nest_f(q, g_\mu, \{(j-1,g)\}),\\
&\qquad z_{g_1,1}, \ldots, z_{g_\mu,s_\mu})\\
&\quad [u \leftarrow u_i,\\
&\quad y'_1, \ldots, y'_q \leftarrow sub(\phi, \pi 2), \ldots, sub(\phi, \pi(q+1)),\\
&\quad z_{g_1,1}, \ldots, z_{g_\mu,s_\mu} \leftarrow par_\phi(\pi, g_1, 1), \ldots, par_\phi(\pi, g_\mu, s_\mu)])\\
&[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p])
\end{aligned}
$$

= (by reordering of normal form computation and substitutions)

$$
\begin{aligned}
&nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair},\\
&\quad ((j-1)_f, l_g)(u, nest_f(1, g_1, \{(j-1,g)\})\\
&\qquad\qquad [y'_b \leftarrow sub(\phi, \pi(b+1)), b \in [q]],\\
&\qquad \ldots,\\
&\qquad nest_f(q, g_\mu, \{(j-1,g)\})\\
&\qquad\qquad [y'_b \leftarrow sub(\phi, \pi(b+1)), b \in [q]],\\
&\qquad z_{g_1,1}, \ldots, z_{g_\mu,s_\mu})\\
&\quad [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]\\
&\quad [z_{g',m} \leftarrow par_\phi(\pi, g', m)[u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p],\\
&\qquad g' \in G^{(s'+1)}, m \in [s']]))
\end{aligned}
$$

= (by reordering of substitutions and normal form computations)

$$
\begin{aligned}
&nf(\Rightarrow_{R_{1;2}}, ((j-1)_f, l_g)(t_i, y_{1,g_1}, \ldots, y_{q,g_\mu}, z_{g_1,1}, \ldots, z_{g_\mu,s_\mu}))\\
&\quad [y_{k,g'} \leftarrow nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, nest_f(k, g', \{(j-1,g)\})\\
&\qquad\qquad [y'_b \leftarrow sub(\phi, \pi(b+1)), b \in [q]])\\
&\qquad [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]),\\
&\qquad k \in [q], g' \in G^{(s'+1)}]\\
&\quad [z_{g',m} \leftarrow nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, par_\phi(\pi, g', m))\\
&\qquad [u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]),\\
&\qquad g' \in G^{(s'+1)}, m \in [s']].
\end{aligned}
$$

From $\vdash_{\phi,(t_1,\ldots,t_p)} (\pi j, g)$ follows by Lemma A.14(2a) that

$$
\vdash_{f,t_i} (j-1, g)\,,
$$

hence applying the induction hypothesis **I**b for t_i gives the following equivalent expression:

$$
\begin{aligned}
&nf(\Rightarrow_{Pre}, \overline{par}_{f,t_i}(j-1, g, l))\\
&\quad [y_{k,g'} \leftarrow nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, nest_f(k, g', \{(j-1,g)\})\\
&\qquad\qquad [y'_b \leftarrow sub(\phi, \pi(b+1)), b \in [q]])\\
&\qquad [u \leftarrow u_i][u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]),\\
&\qquad k \in [q], g' \in G^{(s'+1)}]\\
&\quad [z_{g',m} \leftarrow nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, par_\phi(\pi, g', m))\\
&\qquad [u_1, \ldots, u_p \leftarrow t_1, \ldots, t_p]),\\
&\qquad g' \in G^{(s'+1)}, m \in [s']].
\end{aligned}
$$

Now, we want to apply—to the normal forms replaced for the $y_{k,g'}$—the previously proven statement **II**(a)**ii**(B) for the

$$sub(\phi,\pi 2),\ldots,sub(\phi,\pi(q+1)) \in RHS(F,\Delta,U_p,Y_r)$$

and $\mathcal{C}=\{(j-1,g)\}$. In order to do so, we must establish that for every variable $y_{k,g'}$ occurring in $nf(\Rightarrow_{Pre},\overline{par}_{f,t_i}(j-1,g,l))$ we have:

$$\vdash_{f,t_i}(k,g') \text{ and not } (k,g') \prec^*_{f,t_i} \{(j-1,g)\}.$$

For every such a $y_{k,g'}$ we have $(j-1,g)\prec_{f,t_i}(k,g')$ by Lemma A.9(2). Hence, $\vdash_{f,t_i}(k,g')$ follows from the above established $\vdash_{f,t_i}(j-1,g)$ by Lemma A.7(1). Furthermore, if $(k,g')\prec^*_{f,t_i}(j-1,g)$ would hold, then we would also have $(j-1,g)\prec^+_{f,t_i}(j-1,g)$, which by Lemma A.8 and $\vdash_{f,t_i}(j-1,g)$ is impossible.
Hence, we can indeed replace the normal forms substituted for the $y_{k,g'}$ according to statement **II**(a)**ii**(B), and obtain:

$$\begin{aligned}
&nf(\Rightarrow_{Pre},\overline{par}_{f,t_i}(j-1,g,l))\\
&\quad[y_{k,g'} \leftarrow nf(\Rightarrow_{R_2},g'(y_k,\overline{par}_{f,t_i}(k,g',1),\ldots,\overline{par}_{f,t_i}(k,g',s'))\\
&\qquad\qquad[y_b\leftarrow nf(\Rightarrow_{R_1},sub(\phi,\pi(b+1))\\
&\qquad\qquad\qquad[u_c\leftarrow t_c,c\in[p]]),b\in[q]]),\\
&\qquad\quad k\in[q],g'\in G^{(s'+1)}]\\
&\quad[z_{g',m}\leftarrow nf(\Rightarrow_{R_{1;2}},nf(\Rightarrow_{R_2\cup Pair},par_\phi(\pi,g',m))\\
&\qquad\qquad[u_1,\ldots,u_p\leftarrow t_1,\ldots,t_p]),\\
&\qquad\quad g'\in G^{(s'+1)},m\in[s']]\\
&\quad=\ (\text{by } \vdash_{f,t_i}(j-1,g) \text{ and Lemma A.11, where for every } b\in[q]:\\
&\qquad\quad \theta_b=nf(\Rightarrow_{R_1},sub(\phi,\pi(b+1))[u_c\leftarrow t_c,c\in[p]])\in T_\Delta(Y_r))\\
&nf(\Rightarrow_{R_2},\overline{par}_{f,t_i}(j-1,g,l)\\
&\qquad\quad[y_b\leftarrow nf(\Rightarrow_{R_1},sub(\phi,\pi(b+1))[u_c\leftarrow t_c,c\in[p]]),b\in[q]])\\
&\quad[z_{g',m}\leftarrow nf(\Rightarrow_{R_{1;2}},nf(\Rightarrow_{R_2\cup Pair},par_\phi(\pi,g',m))\\
&\qquad\qquad[u_1,\ldots,u_p\leftarrow t_1,\ldots,t_p]),\\
&\qquad\quad g'\in G^{(s'+1)},m\in[s']].
\end{aligned}$$

We would like to apply the induction hypothesis for π to the normal forms substituted for the $z_{g',m}$ in the previous expression. In order to do so, we must establish that for every $g'\in G^{(s'+1)}$ and $m\in[s']$, if $z_{g',m}$ occurs in

$$\begin{aligned}
&nf(\Rightarrow_{R_2},\overline{par}_{f,t_i}(j-1,g,l)\\
&\qquad[y_b\leftarrow nf(\Rightarrow_{R_1},sub(\phi,\pi(b+1))[u_c\leftarrow t_c,c\in[p]]),b\in[q]])\,,
\end{aligned}$$

then $\vdash_{\phi,(t_1,\ldots,t_p)}(\pi,g')$. Since such a $z_{g',m}$ would necessarily have to occur in $\overline{par}_{f,t_i}(j-1,g,l)$, this condition follows from the prerequisite $\vdash_{\phi,(t_1,\ldots,t_p)}(\pi j,g)$ by Lemma A.14(2b).
Hence, we obtain the following equivalent expression:

$$\begin{aligned}
&nf(\Rightarrow_{R_2},\overline{par}_{f,t_i}(j-1,g,l)\\
&\qquad\quad[y_b\leftarrow nf(\Rightarrow_{R_1},sub(\phi,\pi(b+1))[u_c\leftarrow t_c,c\in[p]]),b\in[q]])\\
&\quad[z_{g',m}\leftarrow\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi,g',m),\ g'\in G^{(s'+1)},m\in[s']]\\
&\quad=\ (\text{by composition of substitutions and}
\end{aligned}$$

since the $\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g', m)$ are $\Rightarrow_{R_2}$-normal forms)

$$nf(\Rightarrow_{R_2}, \overline{par}_{f,t_i}(j-1, g, l)$$
$$[y_b \leftarrow nf(\Rightarrow_{R_1}, sub(\phi, \pi(b+1))[u_c \leftarrow t_c, c \in [p]]), b \in [q],$$
$$z_{g',m} \leftarrow \overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi, g', m),\ g' \in G^{(s'+1)}, m \in [s']])$$

= (by definition of $\overline{par}_{\phi,(t_1,\ldots,t_p)}$)

$\overline{par}_{\phi,(t_1,\ldots,t_p)}(\pi j, g, l)$. □

Using the previous lemma, we can now prove the correctness of Construction 5.1.

Theorem A.16 (*Theorem 5.2; correctness of Construction 5.1*)
$\tau(M_1); \tau(M_2) = \tau(M_{1;2})$

Proof
For every $t \in T_\Sigma$, we calculate:

$\tau(M_{1;2})(t)$

= (by definition of $\tau(M_{1;2})$)

$nf(\Rightarrow_{R_{1;2}}, e_{1;2}[x \leftarrow t])$

= (by definition of $e_{1;2}$ in Construction 5.1)

$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, e_2[x \leftarrow e_1])[x \leftarrow t])$

= (by substitution)

$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, e_2[x \leftarrow e_1][x \leftarrow u_1])[u_1 \leftarrow t])$

= (by substitution)

$nf(\Rightarrow_{R_{1;2}}, nf(\Rightarrow_{R_2 \cup Pair}, e_2[x \leftarrow v_1][v_1 \leftarrow e_1[x \leftarrow u_1]])[u_1 \leftarrow t])$

= (by statement **II**(a)**ii**(A) of Lemma A.15 for $p = 1$, $t_1 = t \in T_\Sigma$, $r = 0$,
$q = 1$, $\phi_1 = e_1[x \leftarrow u_1] \in RHS(F, \Delta, \{u_1\}, \emptyset)$, $s = 0$
and $\psi = e_2[x \leftarrow v_1] \in RHS(G, \Omega, \{v_1\}, \emptyset)$)

$nf(\Rightarrow_{R_2}, e_2[x \leftarrow v_1][v_1 \leftarrow nf(\Rightarrow_{R_1}, e_1[x \leftarrow u_1][u_1 \leftarrow t])])$

= (by substitution)

$nf(\Rightarrow_{R_2}, e_2[x \leftarrow nf(\Rightarrow_{R_1}, e_1[x \leftarrow t])])$

= (by definition of $\tau(M_1)$)

$nf(\Rightarrow_{R_2}, e_2[x \leftarrow \tau(M_1)(t)])$

= (by definition of $\tau(M_2)$)

$\tau(M_2)(\tau(M_1)(t))$. □

References

Engelfriet, J. (1981). *Tree transducers and syntax directed semantics*. Tech. rept. 363. Technische Hogeschool Twente.

Engelfriet, J., & Vogler, H. (1985). Macro tree transducers. *J. Comput. Syst. Sci.*, **31**, 71–145.

Fülöp, Z., & Vogler, H. (1998). *Syntax-Directed Semantics — Formal Models Based on Tree Transducers*. Monographs in Theoretical Computer Science, An EATCS Series. Springer-Verlag.

Kühnemann, A., & Vogler, H. (1994). Synthesized and inherited functions — a new computational model for syntax-directed semantics. *Acta Informatica*, **31**, 431–477.