Free Theorems — Foundations

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Using a Free Theorem [Wadler 1989]

For every

$$\texttt{get} :: [\alpha] \to [\alpha]$$

we have

$$\texttt{map}\ f\ (\texttt{get}\ l) \quad = \quad \texttt{get}\ (\texttt{map}\ f\ l)$$

for arbitrary f and l, where

$$\begin{array}{l} \texttt{map} :: (\alpha \to \beta) \to [\alpha] \to [\beta] \\ \texttt{map}\ f\ [\,] \qquad\quad = [\,] \\ \texttt{map}\ f\ (a : as) = (f\ a) : (\texttt{map}\ f\ as) \end{array}$$

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But how do we know this?

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- `get` $:: [\alpha] \rightarrow [\alpha]$ must work uniformly for every instantiation of α.

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- That is what was claimed!

Another Example

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\begin{aligned}
&\texttt{takeWhile} :: (\alpha \to \mathsf{Bool}) \to [\alpha] \to [\alpha] \\
&\texttt{takeWhile}\ p\ []\ = [] \\
&\texttt{takeWhile}\ p\ (a : as)\ \mid\ p\ a = a : (\texttt{takeWhile}\ p\ as) \\
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For arbitrary p, f, and l:

$$\texttt{takeWhile}\ p\ (\texttt{map}\ f\ l) \quad = \quad \texttt{map}\ f\ (\texttt{takeWhile}\ (p \circ f)\ l)$$

Provable by induction.

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$$\texttt{takeWhile} :: (\alpha \to \text{Bool}) \to [\alpha] \to [\alpha]$$

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Automatic Generation of Free Theorems

At `http://linux.tcs.inf.tu-dresden.de/~voigt/ft`:

This tool allows to generate free theorems for sublanguages of Haskell as described here.

The source code of the underlying library and a shell-based application using it is available here and here.

Please enter a (polymorphic) type, e.g. "(a -> Bool) -> [a] -> [a]" or simply "filter":

```
g :: (a -> Bool) -> [a] -> [a]
```

Please choose a sublanguage of Haskell:

- (●) no bottoms (hence no general recursion and no selective strictness)
- ( ) general recursion but no selective strictness
- ( ) general recursion and selective strictness

Please choose a theorem style (without effect in the sublanguage with no bottoms):

- (●) equational
- ( ) inequational

[Generate]

Automatic Generation of Free Theorems

The theorem generated for functions of the type

```
g :: forall a . (a -> Bool) -> [a] -> [a]
```

in the sublanguage of Haskell with no bottoms is:

```
forall t1,t2 in TYPES, R in REL(t1,t2).
 forall p :: t1 -> Bool.
  forall q :: t2 -> Bool.
   (forall (x, y) in R. p x = q y)
   ==> (forall (z, v) in lift{[]}(R).
         (g p z, g q v) in lift{[]}(R))
```

The structural lifting occurring therein is defined as follows:

```
lift{[]}(R)
  = {([], [])}
  u {(x : xs, y : ys) |
       ((x, y) in R) && ((xs, ys) in lift{[]}(R))}
```

Reducing all permissible relation variables to functions yields:

```
forall t1,t2 in TYPES, f :: t1 -> t2.
 forall p :: t1 -> Bool.
  forall q :: t2 -> Bool.
   (forall x :: t1. p x = q (f x))
   ==> (forall y :: [t1]. map f (g p y) = g q (map f y))
```

Export as PDF   Show type instantiations   Enter a new type   Help page

Formal Background: Parametric Polymorphism

Question: What g have type $\forall\alpha.\ (\alpha \rightarrow \mathsf{Bool}) \rightarrow [\alpha] \rightarrow [\alpha]$?

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[\![\mathsf{Bool}]\!] &= \{\mathsf{True}, \mathsf{False}\} \\
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- $[\![\forall\alpha.\tau]\!] = \{g \mid \forall\tau'.\ g_{\tau'} \in [\![\tau[\tau'/\alpha]]\!]\}$?
- But this includes "ad-hoc polymorphic" functions!

Unwanted Ad-Hoc Polymorphism: Example

- With the proposed definition,
  $[\![\forall\alpha.\ (\alpha,\alpha) \to \alpha]\!] = \{g \mid \forall\tau.\ g_\tau : [\![\tau]\!] \times [\![\tau]\!] \to [\![\tau]\!]\}$.

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- But this also allows a g with

$$\begin{array}{lcl} g_{\mathsf{Bool}}\ (x,y) & = & \texttt{not}\ x \\ g_{\mathsf{Int}}\ \ (x,y) & = & y+1\,, \end{array}$$

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- To prevent this, we have to compare

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  But how?

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- Choose a relation $\mathcal{R} \subseteq [\![\mathsf{Bool}]\!] \times [\![\mathsf{Int}]\!]$.
- Call $(x_1, x_2) \in [\![\mathsf{Bool}]\!] \times [\![\mathsf{Bool}]\!]$ and $(y_1, y_2) \in [\![\mathsf{Int}]\!] \times [\![\mathsf{Int}]\!]$ related if $(x_1, y_1) \in \mathcal{R}$ and $(x_2, y_2) \in \mathcal{R}$.

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Reynolds: $g \in [\![\forall\alpha.\tau]\!]$ iff for every τ_1, τ_2 and $\mathcal{R} \subseteq [\![\tau_1]\!] \times [\![\tau_2]\!]$,
g_{τ_1} is related to g_{τ_2} by the "propagation" of $\mathcal{R}$
along τ.

Polymorphic Lambda Calculus [Girard 1972, Reynolds 1974]

Types: $\tau := \alpha \mid \tau \rightarrow \tau \mid \forall\alpha.\tau$

Terms: $t := x \mid \lambda x : \tau.t \mid t\ t \mid \Lambda\alpha.t \mid t\ \tau$

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Parametricity Theorem [Reynolds 1983, Wadler 1989]

Given τ and environments θ_1, θ_2, ρ with $\rho(\alpha) \subseteq \theta_1(\alpha) \times \theta_2(\alpha)$, define $\Delta_{\tau,\rho} \subseteq [\![\tau]\!]_{\theta_1} \times [\![\tau]\!]_{\theta_2}$ as follows:

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Then, for every closed term t of closed type τ:

$$([\![t]\!]_{\emptyset,\emptyset}, [\![t]\!]_{\emptyset,\emptyset}) \in \Delta_{\tau,\emptyset}.$$

Proof Sketch

Prove the following more general statement:

$\Gamma \vdash t : \tau$ implies $(\llbracket t \rrbracket_{\theta_1,\sigma_1}, \llbracket t \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\tau,\rho}$,
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$$\frac{(\llbracket t \rrbracket_{\theta_1,\sigma_1}, \llbracket t \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\tau_1 \to \tau_2,\rho} \qquad (\llbracket u \rrbracket_{\theta_1,\sigma_1}, \llbracket u \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\tau_1,\rho}}{(\llbracket t\ u \rrbracket_{\theta_1,\sigma_1}, \llbracket t\ u \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\tau_2,\rho}}$$

$$\frac{\forall \mathcal{R} \subseteq S_1 \times S_2.\ (\llbracket t \rrbracket_{\theta_1[\alpha \mapsto S_1],\sigma_1}, \llbracket t \rrbracket_{\theta_2[\alpha \mapsto S_2],\sigma_2}) \in \Delta_{\tau,\rho[\alpha \mapsto \mathcal{R}]}}{(\llbracket \Lambda \alpha . t \rrbracket_{\theta_1,\sigma_1}, \llbracket \Lambda \alpha . t \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\forall \alpha . \tau,\rho}}$$

$$\frac{\Gamma \vdash t : \forall \alpha . \tau}{(\llbracket t\ \tau' \rrbracket_{\theta_1,\sigma_1}, \llbracket t\ \tau' \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\tau[\tau'/\alpha],\rho}}$$

Proof Sketch

Prove the following more general statement:

$\Gamma \vdash t : \tau$ implies $(\llbracket t \rrbracket_{\theta_1,\sigma_1}, \llbracket t \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\tau,\rho}$,
provided $(\sigma_1(x), \sigma_2(x)) \in \Delta_{\tau',\rho}$ for every $x : \tau'$ in Γ

by induction on the structure of typing derivations.
The base case is immediate. In the step cases:

$$\frac{\forall (a_1, a_2) \in \Delta_{\tau_1,\rho}.\ (\llbracket t \rrbracket_{\theta_1,\sigma_1[x\mapsto a_1]}, \llbracket t \rrbracket_{\theta_2,\sigma_2[x\mapsto a_2]}) \in \Delta_{\tau_2,\rho}}{(\llbracket \lambda x : \tau_1.t \rrbracket_{\theta_1,\sigma_1}, \llbracket \lambda x : \tau_1.t \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\tau_1\to\tau_2,\rho}}$$

$$\frac{(\llbracket t \rrbracket_{\theta_1,\sigma_1}, \llbracket t \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\tau_1\to\tau_2,\rho} \qquad (\llbracket u \rrbracket_{\theta_1,\sigma_1}, \llbracket u \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\tau_1,\rho}}{(\llbracket t\ u \rrbracket_{\theta_1,\sigma_1}, \llbracket t\ u \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\tau_2,\rho}}$$

$$\frac{\forall \mathcal{R} \subseteq S_1 \times S_2.\ (\llbracket t \rrbracket_{\theta_1[\alpha\mapsto S_1],\sigma_1}, \llbracket t \rrbracket_{\theta_2[\alpha\mapsto S_2],\sigma_2}) \in \Delta_{\tau,\rho[\alpha\mapsto\mathcal{R}]}}{(\llbracket \Lambda\alpha.t \rrbracket_{\theta_1,\sigma_1}, \llbracket \Lambda\alpha.t \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\forall\alpha.\tau,\rho}}$$

$$\frac{(\llbracket t \rrbracket_{\theta_1,\sigma_1}, \llbracket t \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\forall\alpha.\tau,\rho}}{(\llbracket t\ \tau' \rrbracket_{\theta_1,\sigma_1}, \llbracket t\ \tau' \rrbracket_{\theta_2,\sigma_2}) \in \Delta_{\tau[\tau'/\alpha],\rho}}$$

Adding Datatypes

Types: $\tau := \ \cdots \mid \mathsf{Bool} \mid [\tau]$

Terms: $t := \ \cdots \mid \mathsf{True} \mid \mathsf{False} \mid [\,]_\tau \mid t : t \mid \textbf{case } t \textbf{ of } \{\cdots\}$

Adding Datatypes

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$$\Gamma \vdash \mathsf{True} : \mathsf{Bool} \quad , \quad \Gamma \vdash \mathsf{False} : \mathsf{Bool} \quad , \quad \Gamma \vdash [\,]_\tau : [\tau]$$

$$\frac{\Gamma \vdash t : \tau \qquad \Gamma \vdash u : [\tau]}{\Gamma \vdash (t : u) : [\tau]}$$

$$\frac{\Gamma \vdash t : \mathsf{Bool} \qquad \Gamma \vdash u : \tau \qquad \Gamma \vdash v : \tau}{\Gamma \vdash (\textbf{case}\ t\ \textbf{of}\ \{\mathsf{True} \to u\,; \mathsf{False} \to v\}) : \tau}$$

$$\frac{\Gamma \vdash t : [\tau'] \qquad \Gamma \vdash u : \tau \qquad \Gamma, x_1 : \tau', x_2 : [\tau'] \vdash v : \tau}{\Gamma \vdash (\textbf{case}\ t\ \textbf{of}\ \{[] \to u\,; (x_1 : x_2) \to v\}) : \tau}$$

Adding Datatypes

Types: $\tau := \;\cdots \mid \mathsf{Bool} \mid [\tau]$

Terms: $t := \;\cdots \mid \mathsf{True} \mid \mathsf{False} \mid [\,]_\tau \mid t : t \mid \textbf{case } t \textbf{ of } \{\cdots\}$

$$\Gamma \vdash \mathsf{True} : \mathsf{Bool} \quad , \quad \Gamma \vdash \mathsf{False} : \mathsf{Bool} \quad , \quad \Gamma \vdash [\,]_\tau : [\tau]$$

$$\frac{\Gamma \vdash t : \tau \qquad \Gamma \vdash u : [\tau]}{\Gamma \vdash (t : u) : [\tau]}$$

$$\frac{\Gamma \vdash t : \mathsf{Bool} \qquad \Gamma \vdash u : \tau \qquad \Gamma \vdash v : \tau}{\Gamma \vdash (\textbf{case } t \textbf{ of } \{\mathsf{True} \to u\,; \mathsf{False} \to v\}) : \tau}$$

$$\frac{\Gamma \vdash t : [\tau'] \qquad \Gamma \vdash u : \tau \qquad \Gamma, x_1 : \tau', x_2 : [\tau'] \vdash v : \tau}{\Gamma \vdash (\textbf{case } t \textbf{ of } \{[\,] \to u\,; (x_1 : x_2) \to v\}) : \tau}$$

With the straightforward extension of the semantics and with

$$\begin{aligned} \Delta_{\mathsf{Bool},\rho} &= \{(\mathsf{True},\mathsf{True}),(\mathsf{False},\mathsf{False})\} \\ \Delta_{[\tau],\rho} &= \{([x_1,\ldots,x_n],[y_1,\ldots,y_n]) \mid n \geq 0, (x_i,y_i) \in \Delta_{\tau,\rho}\}\,, \end{aligned}$$

the parametricity theorem still holds.

Now Formal Counterpart to Intuitive Reasoning

Given g of type $\forall\alpha.\,(\alpha \to \mathsf{Bool}) \to ([\alpha] \to [\alpha])$,
by the parametricity theorem:

$$(\mathrm{g}, \mathrm{g}) \in \Delta_{\forall\alpha.\,(\alpha\to\mathsf{Bool})\to([\alpha]\to[\alpha]),\emptyset}$$

Now Formal Counterpart to Intuitive Reasoning

Given g of type $\forall\alpha.\ (\alpha \to \mathsf{Bool}) \to ([\alpha] \to [\alpha])$, by the parametricity theorem:

$$
\begin{aligned}
& (\mathrm{g}, \mathrm{g}) \in \Delta_{\forall\alpha.\,(\alpha\to\mathsf{Bool})\to([\alpha]\to[\alpha]),\emptyset} \\
\Leftrightarrow\ & \forall\mathcal{R} \in \mathit{Rel}.\ (\mathrm{g}, \mathrm{g}) \in \Delta_{(\alpha\to\mathsf{Bool})\to([\alpha]\to[\alpha]),[\alpha\mapsto\mathcal{R}]} \\
& \text{by definition of } \Delta
\end{aligned}
$$

Now Formal Counterpart to Intuitive Reasoning

Given g of type $\forall\alpha.\ (\alpha \to \mathsf{Bool}) \to ([\alpha] \to [\alpha])$,
by the parametricity theorem:

$$(\mathrm{g}, \mathrm{g}) \in \Delta_{\forall\alpha.\,(\alpha\to\mathsf{Bool})\to([\alpha]\to[\alpha]),\emptyset}$$
$$\Leftrightarrow \forall\mathcal{R} \in \mathit{Rel}.\ (\mathrm{g}, \mathrm{g}) \in \Delta_{(\alpha\to\mathsf{Bool})\to([\alpha]\to[\alpha]),[\alpha\mapsto\mathcal{R}]}$$
$$\Leftrightarrow \forall\mathcal{R} \in \mathit{Rel}, (a_1, a_2) \in \Delta_{\alpha\to\mathsf{Bool},[\alpha\mapsto\mathcal{R}]}.\ (\mathrm{g}\ a_1, \mathrm{g}\ a_2) \in \Delta_{[\alpha]\to[\alpha],[\alpha\mapsto\mathcal{R}]}$$

by definition of Δ

Now Formal Counterpart to Intuitive Reasoning

Given g of type $\forall\alpha.\,(\alpha \to \mathsf{Bool}) \to ([\alpha] \to [\alpha])$,
by the parametricity theorem:

$$
\begin{aligned}
& (\mathrm{g}, \mathrm{g}) \in \Delta_{\forall\alpha.\,(\alpha\to\mathsf{Bool})\to([\alpha]\to[\alpha]),\emptyset} \\
\Leftrightarrow\ & \forall\mathcal{R} \in \mathit{Rel}.\ (\mathrm{g}, \mathrm{g}) \in \Delta_{(\alpha\to\mathsf{Bool})\to([\alpha]\to[\alpha]),[\alpha\mapsto\mathcal{R}]} \\
\Leftrightarrow\ & \forall\mathcal{R} \in \mathit{Rel}, (a_1, a_2) \in \Delta_{\alpha\to\mathsf{Bool},[\alpha\mapsto\mathcal{R}]}.\ (\mathrm{g}\ a_1, \mathrm{g}\ a_2) \in \Delta_{[\alpha]\to[\alpha],[\alpha\mapsto\mathcal{R}]} \\
\Leftrightarrow\ & \forall\mathcal{R} \in \mathit{Rel}, (a_1, a_2) \in \Delta_{\alpha\to\mathsf{Bool},[\alpha\mapsto\mathcal{R}]}, (l_1, l_2) \in \Delta_{[\alpha],[\alpha\mapsto\mathcal{R}]}. \\
& \quad (\mathrm{g}\ a_1\ l_1, \mathrm{g}\ a_2\ l_2) \in \Delta_{[\alpha],[\alpha\mapsto\mathcal{R}]}
\end{aligned}
$$

by definition of Δ

Now Formal Counterpart to Intuitive Reasoning

Given g of type $\forall\alpha.\ (\alpha \to \mathsf{Bool}) \to ([\alpha] \to [\alpha])$,
by the parametricity theorem:

$$
\begin{aligned}
& (g, g) \in \Delta_{\forall\alpha.\,(\alpha\to\mathsf{Bool})\to([\alpha]\to[\alpha]),\emptyset} \\
\Leftrightarrow\ & \forall\mathcal{R} \in \mathit{Rel}.\ (g, g) \in \Delta_{(\alpha\to\mathsf{Bool})\to([\alpha]\to[\alpha]),[\alpha\mapsto\mathcal{R}]} \\
\Leftrightarrow\ & \forall\mathcal{R} \in \mathit{Rel}, (a_1, a_2) \in \Delta_{\alpha\to\mathsf{Bool},[\alpha\mapsto\mathcal{R}]}.\ (g\ a_1, g\ a_2) \in \Delta_{[\alpha]\to[\alpha],[\alpha\mapsto\mathcal{R}]} \\
\Leftrightarrow\ & \forall\mathcal{R} \in \mathit{Rel}, (a_1, a_2) \in \Delta_{\alpha\to\mathsf{Bool},[\alpha\mapsto\mathcal{R}]}, (l_1, l_2) \in \Delta_{[\alpha],[\alpha\mapsto\mathcal{R}]}. \\
& \quad (g\ a_1\ l_1, g\ a_2\ l_2) \in \Delta_{[\alpha],[\alpha\mapsto\mathcal{R}]} \\
\Rightarrow\ & \forall(a_1, a_2) \in \Delta_{\alpha\to\mathsf{Bool},[\alpha\mapsto f]}, (l_1, l_2) \in (\mathtt{map}\ f). \\
& \quad (g\ a_1\ l_1, g\ a_2\ l_2) \in (\mathtt{map}\ f)
\end{aligned}
$$

by instantiating $\mathcal{R} = f$ and realising that $\Delta_{[\alpha],[\alpha\mapsto f]} = \mathtt{map}\ f$

for every function f

Now Formal Counterpart to Intuitive Reasoning

Given $\mathtt{g}$ of type $\forall\alpha.\ (\alpha \to \mathsf{Bool}) \to ([\alpha] \to [\alpha])$,
by the parametricity theorem:

$$
\begin{aligned}
& (\mathrm{g}, \mathrm{g}) \in \Delta_{\forall\alpha.\,(\alpha\to\mathsf{Bool})\to([\alpha]\to[\alpha]),\emptyset} \\
\Leftrightarrow\ & \forall\mathcal{R} \in Rel.\ (\mathrm{g}, \mathrm{g}) \in \Delta_{(\alpha\to\mathsf{Bool})\to([\alpha]\to[\alpha]),[\alpha\mapsto\mathcal{R}]} \\
\Leftrightarrow\ & \forall\mathcal{R} \in Rel, (a_1, a_2) \in \Delta_{\alpha\to\mathsf{Bool},[\alpha\mapsto\mathcal{R}]}.\ (\mathrm{g}\ a_1, \mathrm{g}\ a_2) \in \Delta_{[\alpha]\to[\alpha],[\alpha\mapsto\mathcal{R}]} \\
\Leftrightarrow\ & \forall\mathcal{R} \in Rel, (a_1, a_2) \in \Delta_{\alpha\to\mathsf{Bool},[\alpha\mapsto\mathcal{R}]}, (l_1, l_2) \in \Delta_{[\alpha],[\alpha\mapsto\mathcal{R}]}. \\
& \quad (\mathrm{g}\ a_1\ l_1, \mathrm{g}\ a_2\ l_2) \in \Delta_{[\alpha],[\alpha\mapsto\mathcal{R}]} \\
\Rightarrow\ & \forall(a_1, a_2) \in \Delta_{\alpha\to\mathsf{Bool},[\alpha\mapsto f]}, (l_1, l_2) \in (\mathtt{map}\ f). \\
& \quad (\mathtt{g}\ a_1\ l_1, \mathtt{g}\ a_2\ l_2) \in (\mathtt{map}\ f) \\
\Rightarrow\ & \forall(l_1, l_2) \in (\mathtt{map}\ f).\ (\mathtt{g}\ (p \circ f)\ l_1, \mathtt{g}\ p\ l_2) \in (\mathtt{map}\ f) \\
& \text{by instantiating } (a_1, a_2) = (p \circ f, p) \in \Delta_{\alpha\to\mathsf{Bool},[\alpha\mapsto f]}
\end{aligned}
$$

for every function f and predicate p.

Now Formal Counterpart to Intuitive Reasoning

Given g of type $\forall\alpha.\ (\alpha \to \mathsf{Bool}) \to ([\alpha] \to [\alpha])$,
by the parametricity theorem:

$$
\begin{aligned}
& (g, g) \in \Delta_{\forall\alpha.\,(\alpha\to\mathsf{Bool})\to([\alpha]\to[\alpha]),\emptyset} \\
\Leftrightarrow\ & \forall \mathcal{R} \in \mathit{Rel}.\ (g, g) \in \Delta_{(\alpha\to\mathsf{Bool})\to([\alpha]\to[\alpha]),[\alpha\mapsto\mathcal{R}]} \\
\Leftrightarrow\ & \forall \mathcal{R} \in \mathit{Rel}, (a_1, a_2) \in \Delta_{\alpha\to\mathsf{Bool},[\alpha\mapsto\mathcal{R}]}.\ (g\ a_1, g\ a_2) \in \Delta_{[\alpha]\to[\alpha],[\alpha\mapsto\mathcal{R}]} \\
\Leftrightarrow\ & \forall \mathcal{R} \in \mathit{Rel}, (a_1, a_2) \in \Delta_{\alpha\to\mathsf{Bool},[\alpha\mapsto\mathcal{R}]}, (l_1, l_2) \in \Delta_{[\alpha],[\alpha\mapsto\mathcal{R}]}. \\
& \quad (g\ a_1\ l_1, g\ a_2\ l_2) \in \Delta_{[\alpha],[\alpha\mapsto\mathcal{R}]} \\
\Rightarrow\ & \forall (a_1, a_2) \in \Delta_{\alpha\to\mathsf{Bool},[\alpha\mapsto f]}, (l_1, l_2) \in (\mathtt{map}\ f). \\
& \quad (g\ a_1\ l_1, g\ a_2\ l_2) \in (\mathtt{map}\ f) \\
\Rightarrow\ & \forall (l_1, l_2) \in (\mathtt{map}\ f).\ (g\ (p \circ f)\ l_1, g\ p\ l_2) \in (\mathtt{map}\ f) \\
\Leftrightarrow\ & \forall l_1.\ \mathtt{map}\ f\ (g\ (p \circ f)\ l_1) = g\ p\ (\mathtt{map}\ f\ l_1)
\end{aligned}
$$

by inlining

for every function f and predicate p.

Now Formal Counterpart to Intuitive Reasoning

Given g of type $\forall\alpha.\ (\alpha \to \text{Bool}) \to ([\alpha] \to [\alpha])$,
by the parametricity theorem:

$$
\begin{aligned}
& (g, g) \in \Delta_{\forall\alpha.\,(\alpha\to\text{Bool})\to([\alpha]\to[\alpha]),\emptyset} \\
\Leftrightarrow\ & \forall\mathcal{R} \in \mathit{Rel}.\ (g, g) \in \Delta_{(\alpha\to\text{Bool})\to([\alpha]\to[\alpha]),[\alpha\mapsto\mathcal{R}]} \\
\Leftrightarrow\ & \forall\mathcal{R} \in \mathit{Rel}, (a_1, a_2) \in \Delta_{\alpha\to\text{Bool},[\alpha\mapsto\mathcal{R}]}.\ (g\ a_1, g\ a_2) \in \Delta_{[\alpha]\to[\alpha],[\alpha\mapsto\mathcal{R}]} \\
\Leftrightarrow\ & \forall\mathcal{R} \in \mathit{Rel}, (a_1, a_2) \in \Delta_{\alpha\to\text{Bool},[\alpha\mapsto\mathcal{R}]}, (l_1, l_2) \in \Delta_{[\alpha],[\alpha\mapsto\mathcal{R}]}. \\
& \quad (g\ a_1\ l_1, g\ a_2\ l_2) \in \Delta_{[\alpha],[\alpha\mapsto\mathcal{R}]} \\
\Rightarrow\ & \forall (a_1, a_2) \in \Delta_{\alpha\to\text{Bool},[\alpha\mapsto f]}, (l_1, l_2) \in (\texttt{map}\ f). \\
& \quad (g\ a_1\ l_1, g\ a_2\ l_2) \in (\texttt{map}\ f) \\
\Rightarrow\ & \forall (l_1, l_2) \in (\texttt{map}\ f).\ (g\ (p \circ f)\ l_1, g\ p\ l_2) \in (\texttt{map}\ f) \\
\Leftrightarrow\ & \forall l_1.\ \texttt{map}\ f\ (g\ (p \circ f)\ l_1) = g\ p\ (\texttt{map}\ f\ l_1)
\end{aligned}
$$

for every function f and predicate p.

That is what was claimed!

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