Knuth's 0-1-Principle and Beyond

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The Sorting Problem

Task: Given a list and an order on the type of elements of this list, produce a sorted list (with same content)!

Example:

12	7	9	8	4	6	$\mapsto$	4	6	7	8	9	12

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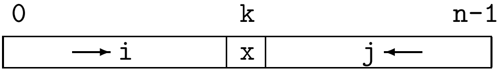

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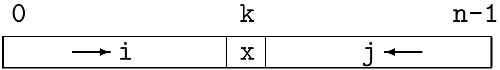

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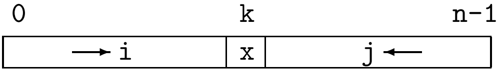

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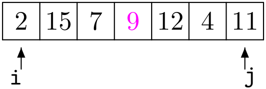

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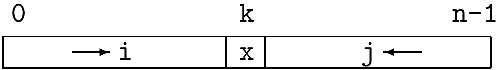

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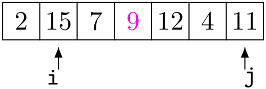

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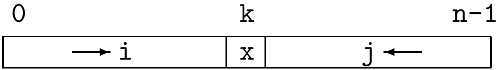

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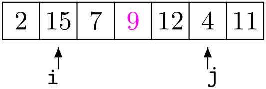

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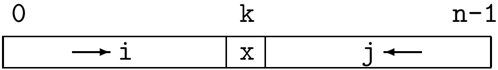

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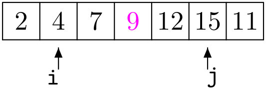

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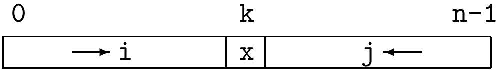

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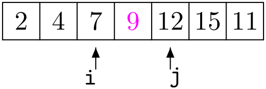

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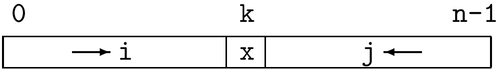

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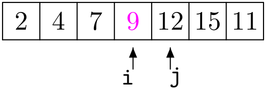

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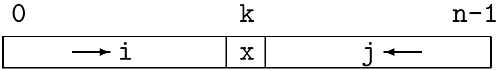

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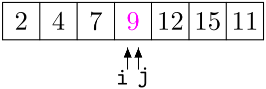

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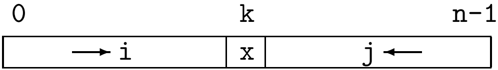

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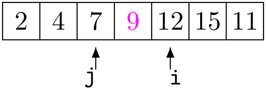

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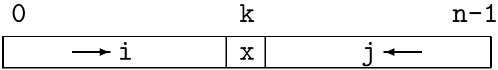

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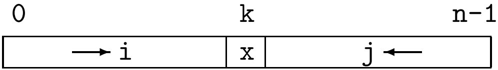

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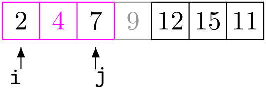

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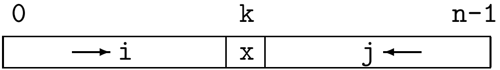

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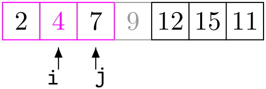

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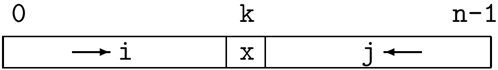

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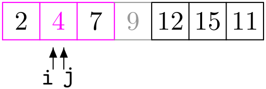

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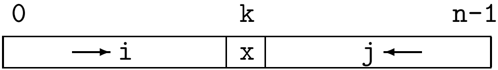

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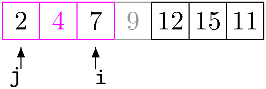

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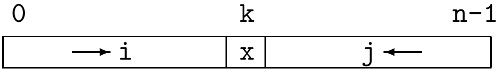

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- works only for lists whose length is a power of two
- complexity is $O(n \cdot \log(n)^2)$
- particularly suitable for hardware and parallel implementations
- correctness is not obvious

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$$f :: (\mathsf{Int}, \mathsf{Int}) \rightarrow (\mathsf{Int}, \mathsf{Int})$$
$$f\ (x, y) = \mathit{if}\ x > y\ \mathit{then}\ (y, x)\ \mathit{else}\ (x, y)$$

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If $\forall xs :: [\mathsf{Bool}], ys = \texttt{sort}\ g\ xs.\ P(xs,ys) \wedge Q(ys)$,
then $\forall xs :: [\mathsf{Int}], ys = \texttt{sort}\ f\ xs.\ P(xs,ys) \wedge Q(ys)$,
where $P(xs,ys)$:= xs and ys contain the same elements in the same multiplicity
$Q(ys)$:= ys is sorted

Using the Free Theorems Generator

Input: `sort::((a,a)->(a,a))->[a]->[a]`

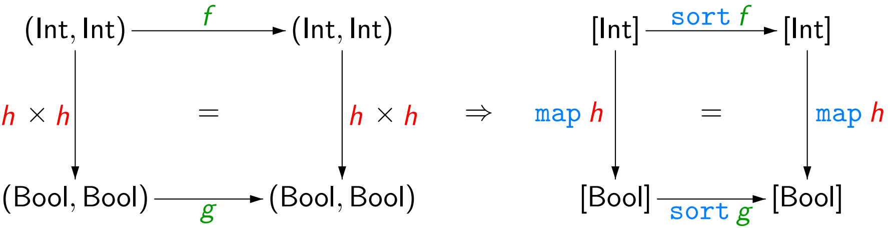
Output:

```
forall t1,t2 in TYPES, h::t1->t2.
 forall f::(t1,t1)->(t1,t1).
  forall g::(t2,t2)->(t2,t2).
   (forall (x,y) in lift_{(,)}(h,h).
     (f x,g y) in lift_{(,)}(h,h))
   ==> (forall xs::[t1].
         map h (sort f xs) = sort g (map h xs))

lift_{(,)}(h,h)
 = {((x1,x2),(y1,y2)) | (h x1 = y1)
                        && (h x2 = y2)}
```

More Specific (and Intuitive)

For every `sort` :: $((\alpha, \alpha) \to (\alpha, \alpha)) \to [\alpha] \to [\alpha]$,
f :: (Int, Int) $\to$ (Int, Int), g :: (Bool, Bool) $\to$ (Bool, Bool), and
h :: Int $\to$ Bool:



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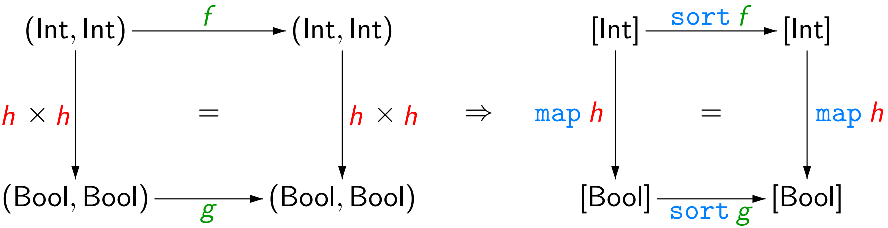
$f :: (\mathsf{Int}, \mathsf{Int}) \to (\mathsf{Int}, \mathsf{Int})$, $g :: (\mathsf{Bool}, \mathsf{Bool}) \to (\mathsf{Bool}, \mathsf{Bool})$, and
$h :: \mathsf{Int} \to \mathsf{Bool}$:



If f and g are as defined before, then the precondition is fulfilled for any h of the form $h\ x = n < x$ for some $n :: \mathsf{Int}$.

Knuth's 0-1-Principle

Informally: If a comparison-swap algorithm sorts Booleans correctly, it sorts integers correctly as well.

Formally: Use Haskell. Let

$$\texttt{sort} :: ((\alpha,\alpha) \to (\alpha,\alpha)) \to [\alpha] \to [\alpha]$$

$$f :: (\mathsf{Int},\mathsf{Int}) \to (\mathsf{Int},\mathsf{Int})$$
$$f\ (x,y) = \mathit{if}\ x > y\ \mathit{then}\ (y,x)\ \mathit{else}\ (x,y)$$

$$g :: (\mathsf{Bool},\mathsf{Bool}) \to (\mathsf{Bool},\mathsf{Bool})$$
$$g\ (x,y) = (x\ \&\&\ y, x \mid\mid y)$$

If $\forall xs :: [\mathsf{Bool}], ys = \texttt{sort}\ g\ xs.\ P(xs,ys) \wedge Q(ys)$,
then $\forall xs :: [\mathsf{Int}], ys = \texttt{sort}\ f\ xs.\ P(xs,ys) \wedge Q(ys)$,
where $P(xs,ys) :=$ xs and ys contain the same elements in the same multiplicity
$Q(ys) :=$ ys is sorted

Proof of "P on [Bool] implies P on [Int]"

Recall: $P(xs, ys) :=$ *xs* and *ys* contain the same elements in the same multiplicity

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If for every $xs :: [\mathsf{Bool}]$, $\texttt{sort}\ g\ xs$ gives the correct result, then for every $xs :: [\mathsf{Int}]$, $\texttt{sort}\ f\ xs$ gives the correct result.

And Beyond?

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- Free theorems allow for a particularly elegant proof of this principle. (This was not my idea: [Day et al. 1999]!)
- Can we do something similar for other algorithm classes?
- Good candidates: algorithms parametrised over some operation, like *cswap* $:: (\alpha, \alpha) \rightarrow (\alpha, \alpha)$ in the case of sorting.

Parallel Prefix Computation

Given: inputs $x_1, \dots, x_n$ and an associative operation $\oplus$

Task: compute the values $x_1, x_1 \oplus x_2, \dots, x_1 \oplus x_2 \oplus \cdots \oplus x_n$

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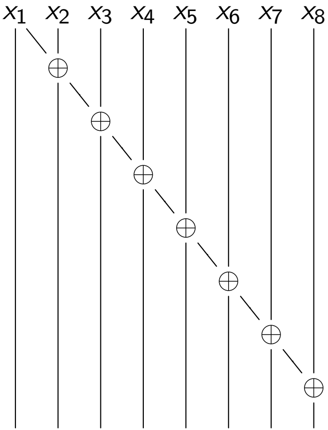
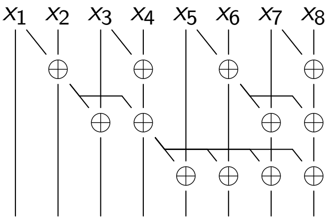
Solution:



Parallel Prefix Computation

Alternative:



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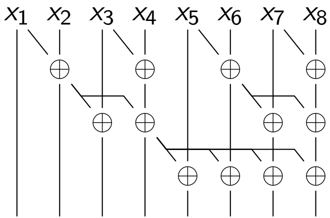
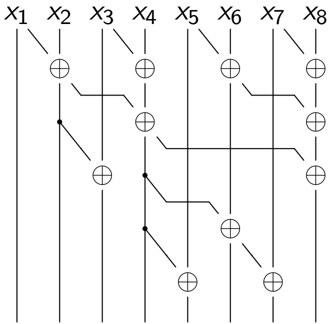


Or:



Or: . . .

In Haskell

Functions of type:

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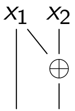
For example, à la [Sklansky 1960]:

$$
\begin{aligned}
&\texttt{sklansky} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha] \\
&\texttt{sklansky}\ (\oplus)\ [x] = [x] \\
&\texttt{sklansky}\ (\oplus)\ \mathit{xs}\ = \mathit{us} +\!\!+\ \mathit{vs} \\
&\quad \textbf{where}\ t \qquad\quad = ((\texttt{length}\ \mathit{xs}) + 1)\ \text{`}\texttt{div}\text{`}\ 2 \\
&\qquad\qquad (\mathit{ys}, \mathit{zs}) = \texttt{splitAt}\ t\ \mathit{xs} \\
&\qquad\qquad \mathit{us} \qquad\ = \texttt{sklansky}\ (\oplus)\ \mathit{ys} \\
&\qquad\qquad \mathit{vs} \qquad\ = [(\texttt{last}\ \mathit{us}) \oplus v \mid v \leftarrow \texttt{sklansky}\ (\oplus)\ \mathit{zs}]
\end{aligned}
$$

Sklansky's Method Visualised



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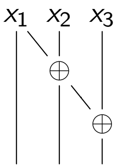

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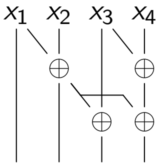

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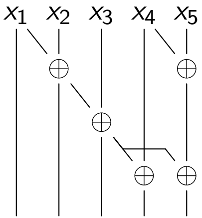

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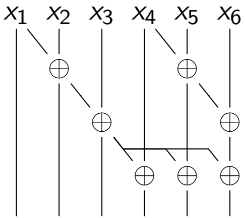

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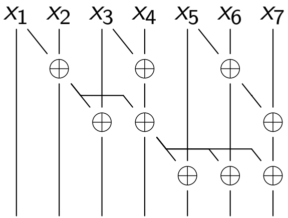

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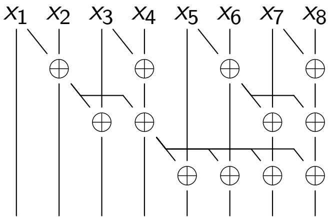

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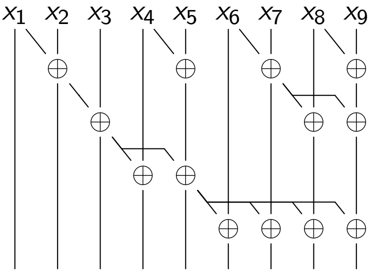

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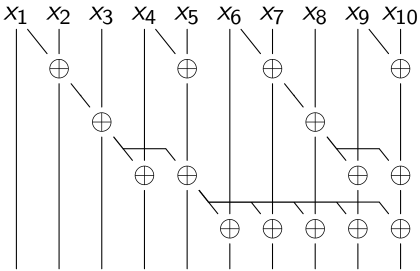

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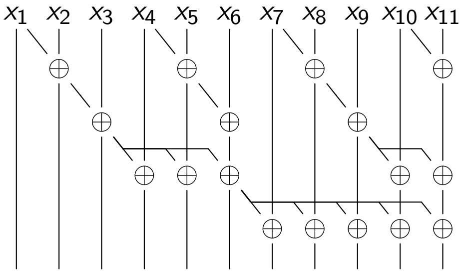

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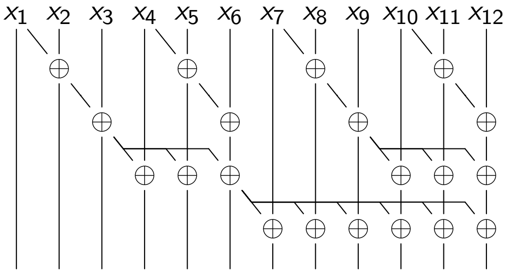

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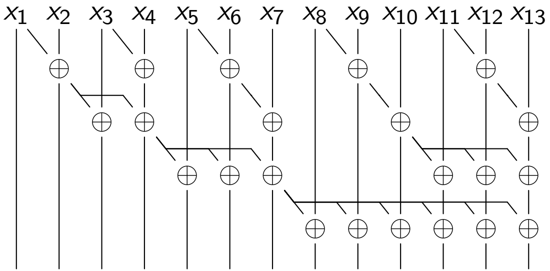

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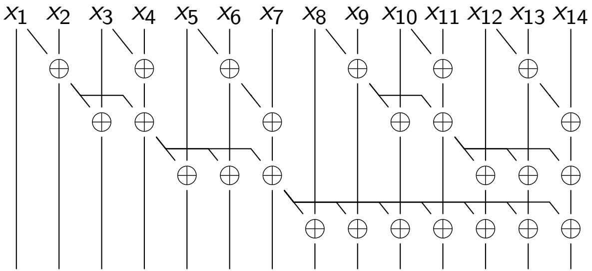

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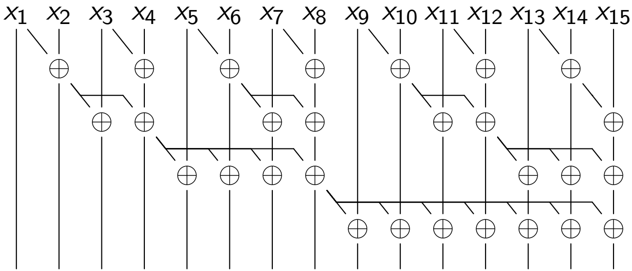

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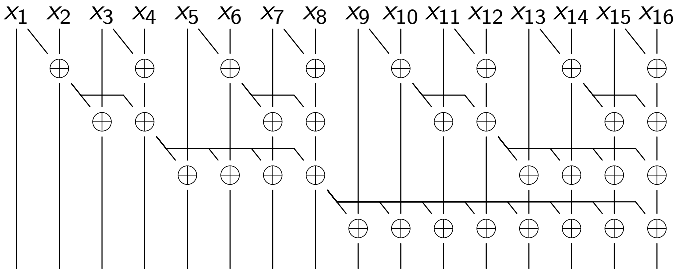

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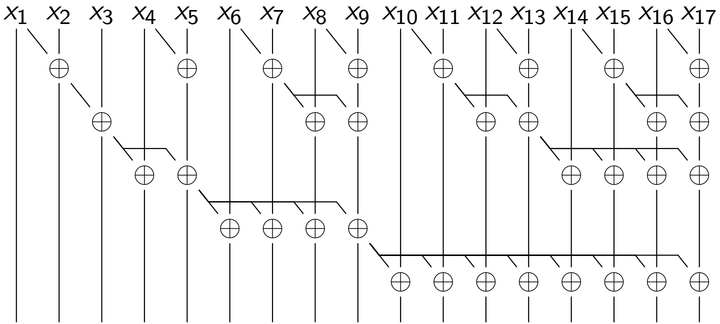

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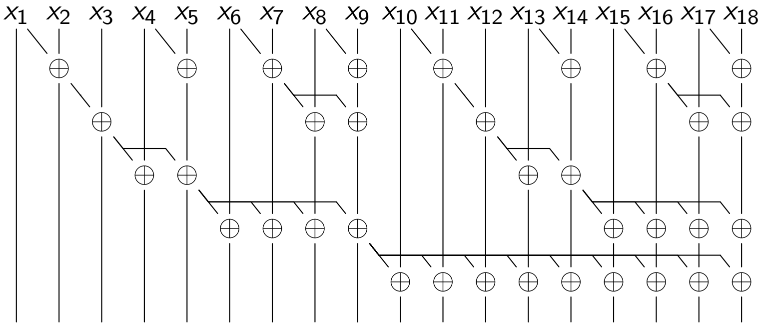

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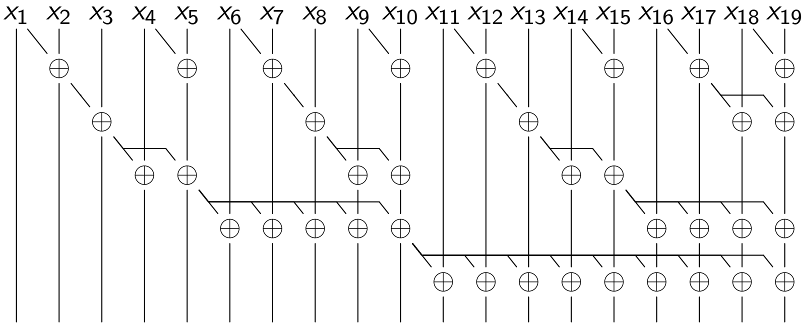

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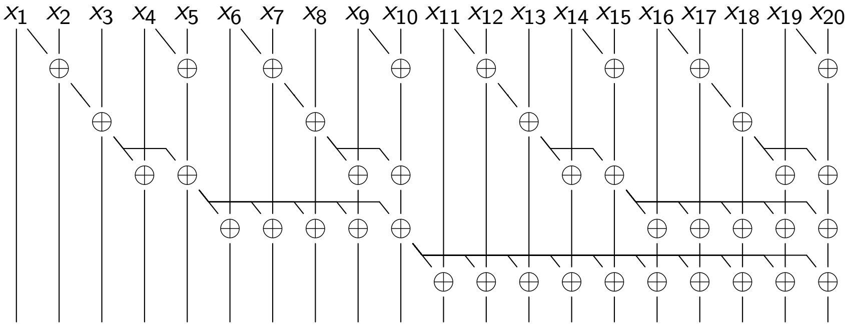

Investigating Particular Instances Only

Knuth's 0-1-Principle

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If a parallel prefix algorithm is correct (for associative operations) on the Booleans, it is so on arbitrary value sets.

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Unfortunately not !

A Knuth-like 0-1-2-Principle [V. 2008]

Given: $\texttt{scanl1} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$

$$
\begin{aligned}
&\texttt{scanl1}\ (\oplus)\ (x : xs) = go\ x\ xs \\
&\qquad \textbf{where}\ go\ x\ [\,] \qquad\ \ = [x] \\
&\qquad\qquad\quad go\ x\ (y : ys) = x : (go\ (x \oplus y)\ ys)
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data Three = Zero | One | Two

Theorem: If for every xs :: [Three] and associative $(\oplus)$ :: Three $\to$ Three $\to$ Three,

$$\texttt{candidate}\ (\oplus)\ xs \quad = \quad \texttt{scanl1}\ (\oplus)\ xs\,,$$

then the same holds for every type τ, $xs :: [\tau]$, and associative $(\oplus) :: \tau \to \tau \to \tau$.

Why 0-1-2? And How?

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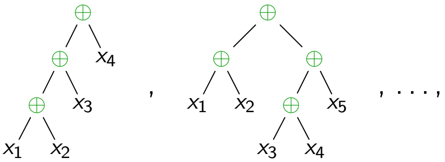

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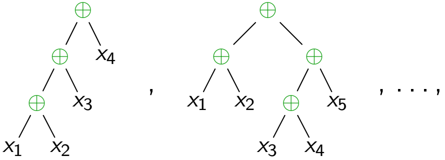

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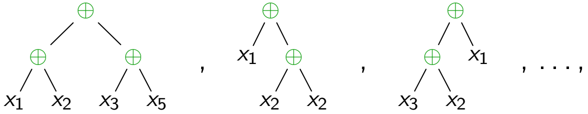

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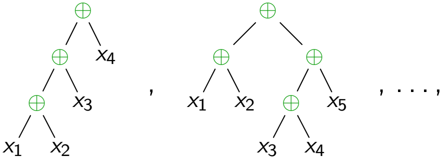

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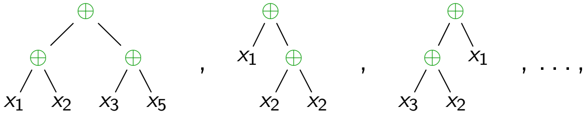

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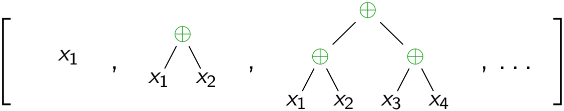

That's How!

Let

$\oplus_1$	Zero	One	Two
Zero	Zero	One	Two
One	One	Two	Two
Two	Two	Two	Two

and

$\oplus_2$	Zero	One	Two
Zero	Zero	One	Two
One	One	One	Two
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A Knuth-like 0-1-2-Principle [V. 2008]

Given: $\texttt{scanl1} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$

$$\begin{aligned}
&\texttt{scanl1}\ (\oplus)\ (x : xs) = go\ x\ xs \\
&\quad \textbf{where}\ go\ x\ [\,] \qquad\quad = [x] \\
&\qquad\qquad\ \ go\ x\ (y : ys) = x : (go\ (x \oplus y)\ ys)
\end{aligned}$$

$\texttt{candidate} :: (\alpha \to \alpha \to \alpha) \to [\alpha] \to [\alpha]$

data Three = Zero | One | Two

Theorem: If for every xs :: [Three] and associative $(\oplus)$ :: Three $\to$ Three $\to$ Three,

$$\texttt{candidate}\ (\oplus)\ xs \quad = \quad \texttt{scanl1}\ (\oplus)\ xs\,,$$

then the same holds for every type τ, $xs :: [\tau]$, and associative $(\oplus) :: \tau \to \tau \to \tau$.

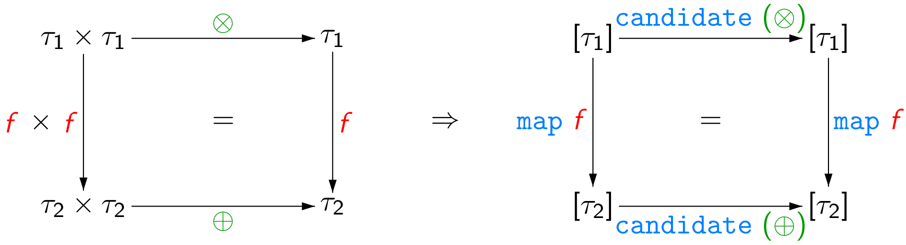
Using the Free Theorems Generator

Input: `candidate :: (a -> a -> a) -> [a] -> [a]`

Output:
```
forall t1,t2 in TYPES, f :: t1 -> t2.
  forall g :: t1 -> t1 -> t1.
    forall h :: t2 -> t2 -> t2.
      (forall x :: t1. forall y :: t1.
         f (g x y) = h (f x) (f y))
      ==> (forall z :: [t1].
                map f (candidate g z)
                 = candidate h (map f z))
```

Rephrased

For every choice of types τ_1, τ_2 and functions $f :: \tau_1 \to \tau_2$, $(\otimes) :: \tau_1 \to \tau_1 \to \tau_1$, and $(\oplus) :: \tau_2 \to \tau_2 \to \tau_2$:



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Decomposing the 0-1-2-Principle

Proposition 1: If `candidate` $(\oplus_1)$ is correct on each list of the form $[(\text{Zero},)^* \ \text{One}\ (,\text{Zero})^* \ (,\text{Two})^*]$ and `candidate` $(\oplus_2)$ is correct on each list of the form $[(\text{Zero},)^* \ \text{One}, \text{Two}\ (,\text{Zero})^*]$, then for every $n \geq 0$,
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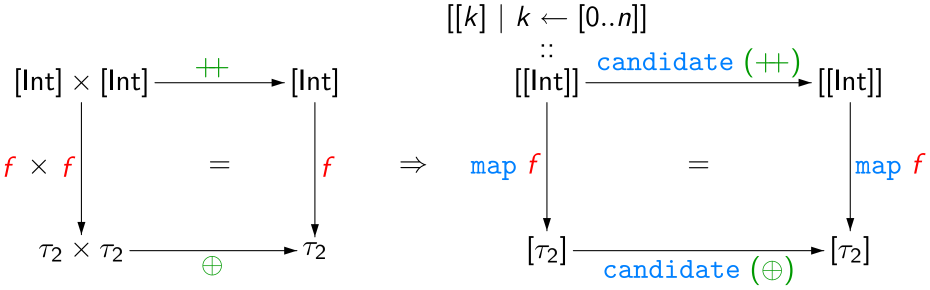

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- For which other algorithm classes can one play the same trick?

References I

G.E. Blelloch.
Prefix sums and their applications.
In J.H. Reif, editor, *Synthesis of Parallel Algorithms*, pages 35–60. Morgan Kaufmann, 1993.

S. Böhme.
Much Ado about Two. Formal proof development.
In *The Archive of Formal Proofs*. http://afp.sf.net/entries/MuchAdoAboutTwo.shtml, 2007.

A. Bove and T. Coquand.
Formalising bitonic sort in type theory.
In *Types for Proofs and Programs, TYPES 2004, Revised Selected Papers*, volume 3839 of *LNCS*, pages 82–97. Springer-Verlag, 2006.

References II

N.A. Day, J. Launchbury, and J.R. Lewis.
Logical abstractions in Haskell.
In *Haskell Workshop, Proceedings*, 1999.

P. Dybjer, Q. Haiyan, and M. Takeyama.
Verifying Haskell programs by combining testing, model checking and interactive theorem proving.
Information & Software Technology, 46(15):1011–1025, 2004.

D.E. Knuth.
The Art of Computer Programming, volume 3: Sorting and Searching.
Addison-Wesley, 1973.

References III

M. Sheeran.
Searching for prefix networks to fit in a context using a lazy functional programming language.
Hardware Design and Functional Languages, 2007.

J. Sklansky.
Conditional-sum addition logic.
IRE Transactions on Electronic Computers, EC-9(6):226–231, 1960.

J. Voigtländer.
Much ado about two: A pearl on parallel prefix computation.
In *Principles of Programming Languages, Proceedings*, pages 29–35. ACM Press, 2008.